Denoting by α the angle formed by the tangent (in the direction of increasing arc of the curve s) with the positive x-direction, we get

$$\cos \alpha = \frac{dx}{ds},$$

$$\sin \alpha = \frac{dy}{ds}.$$

In polar coordinates,

$$ds = \sqrt{(dr)^2 + (r\,d\varphi)^2} = \sqrt{r^2 + \left(\frac{dr}{d\varphi}\right)^2}\,d\varphi$$

Denoting by β the angle between the radius vector of the point of the curve and the tangent to the curve at this point, we have

$$\cos \beta = \frac{dr}{ds},$$

$$\sin \beta = r\frac{d\varphi}{ds}.$$

2°. Curvature of a curve. The *curvature* K of a curve at one of its points M is the limit of the ratio of the angle between the positive directions of the tangents at the points M and N of the curve *(angle of contingence)* to the length of the arc $\smile{MN} = \Delta s$ when $N \to M$ (Fig. 35), that is,

$$K = \lim_{\Delta s \to 0} \frac{\Delta \alpha}{\Delta s} = \frac{d\alpha}{ds},$$

where α is the angle between the positive directions of the tangent at the point M and the x-axis.

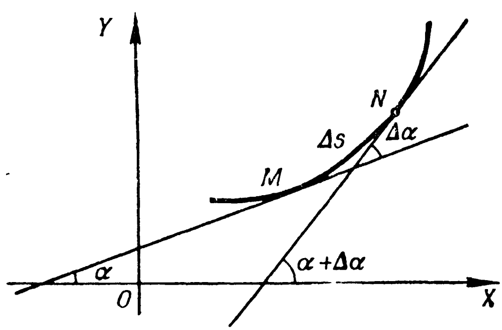

Fig. 35

The *radius of curvature* R is the reciprocal of the absolute value of the curvature, i. e.,

$$R = \frac{1}{|K|}.$$

The circle $\left(K = \frac{1}{a}\right.$, where a is the radius of the circle) and the straight line ($K = 0$) are lines of constant curvature.

We have the following formulas for computing the curvature in rectangular coordinates (accurate to within the sign):

1) if the curve is given by an equation explicitly, $y=f(x)$, then

$$K=\frac{y''}{(1+y'^2)^{3/2}};$$

2) if the curve is given by an equation implicitly, $F(x, y)=0$, then

$$K=\frac{\begin{vmatrix} F''_{xx} & F''_{xy} & F'_x \\ F''_{yx} & F''_{yy} & F'_y \\ F'_x & F'_y & 0 \end{vmatrix}}{\left(F_x'^2+F_y'^2\right)^{3/2}};$$

3) if the curve is represented by equations in parametric form, $x=\varphi(t)$, $y=\psi(t)$, then

$$K=\frac{\begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix}}{(x'^2+y'^2)^{3/2}},$$

where

$$x'=\frac{dx}{dt}, \quad y'=\frac{dy}{dt}, \quad x''=\frac{d^2x}{dt^2}, \quad y''=\frac{d^2y}{dt^2}.$$

In polar coordinates, when the curve is given by the equation $r=f(\varphi)$, we have

$$K=\frac{r^2+2r'^2-rr''}{(r^2+r'^2)^{3/2}},$$

where

$$r'=\frac{dr}{d\varphi} \quad \text{and} \quad r''=\frac{d^2r}{d\varphi^2}.$$

3°. Circle of curvature. The *circle of curvature* (or *osculating circle*) of a curve at the point M is the limiting position of a circle drawn through M and two other points of the curve, P and Q, as $P \longrightarrow M$ and $Q \longrightarrow M$.

The radius of the circle of curvature is equal to the radius of curvature, and the centre of the circle of curvature (the *centre of curvature*) lies on the normal to the curve drawn at the point M in the direction of concavity of the curve.

The coordinates X and Y of the centre of curvature of the curve are computed from the formulas

$$X=x-\frac{y'(1+y'^2)}{y''}, \quad Y=y+\frac{1+y'^2}{y''}.$$

The *evolute* of a curve is the locus of the centres of curvature of the curve.

If in the formulas for determining the coordinates of the centre of curvature we regard X and Y as the current coordinates of a point of the evolute, then these formulas yield parametric equations of the evolute with parameter x or y (or t, if the curve itself is represented by equations in parametric form)

Example 1. Find the equation of the evolute of the parabola $y=x^2$.

Solution. $X = -4x^3$, $Y = \frac{1+6x^2}{2}$. Eliminating the parameter x, we find the equation of the evolute in explicit form, $Y = \frac{1}{2} + 3\left(\frac{X}{4}\right)^{2/3}$.

The *involute* of a curve is a curve for which the given curve is an evolute.

The normal MC of the involute Γ_2 is a tangent to the evolute Γ_1; the length of the arc $\breve{CC_1}$ of the evolute is equal to the corresponding increment in the radius of curvature $\breve{CC_1} = M_1C_1 - MC$; that is why the involute Γ_2 is also called the *evolvent* of the curve Γ_1 obtained by unwinding a taut thread wound onto Γ_1 (Fig. 36). To each evolute there corresponds an infinitude of involutes, which are related to different initial lengths of thread.

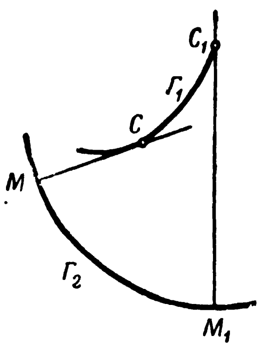

Fig. 36

4°. **Vertices of a curve.** The *vertex* of a curve is a point of the curve at which the curvature has a maximum or a minimum. To determine the vertices of a curve, we form the expression of the curvature K and find its extremal points. In place of the curvature K we can take the radius of curvature $R = \frac{1}{|K|}$ and seek its extremal points if the computations are simpler in this case.

Example 2. Find the vertex of the catenary $y = a \cosh \frac{x}{a}$ $(a > 0)$.

Solution. Since $y' = \sinh \frac{x}{a}$ and $y'' = \frac{1}{a}\cosh\frac{x}{a}$, it follows that $K = \frac{1}{a\cosh^2\frac{x}{a}}$ and, hence, $R = a\cosh^2\frac{x}{a}$. We have $\frac{dR}{dx} = \sinh 2\frac{x}{a}$. Equating the derivative $\frac{dR}{dx}$ to zero, we get $\sinh 2\frac{x}{a} = 0$, whence we find the sole critical point $x = 0$. Computing the second derivative $\frac{d^2R}{dx^2}$ and putting into it the value $x = 0$, we get $\left.\frac{d^2R}{dx^2}\right|_{x=0} = \left.\frac{2}{a}\cosh 2\frac{x}{a}\right|_{x=0} = \frac{2}{a} > 0$. Therefore, $x = 0$ is the minimum point of the radius of curvature (or of the maximum of curvature) of the catenary. The vertex of the catenary $y = a\cosh\frac{x}{a}$ is, thus, the point $A\ (0,\ a)$.

Find the differential of the arc, and also the cosine and sine of the angle formed, with the positive x-direction, by the tangent to each of the following curves:

993. $x^2 + y^2 = a^2$ (circle).

994. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (ellipse).

995 $y^2 = 2px$ (parabola).

996. $x^{2/3}+y^{2/3}=a^{2/3}$ (astroid).

997. $y=a\cosh\frac{x}{a}$ (catenary).

998. $x=a(t-\sin t)$; $y=a(1-\cos t)$ (cycloid).

999. $x=a\cos^3 t$, $y=a\sin^3 t$ (astroid).

Find the differential of the arc, and also the cosine or sine of the angle formed by the radius vector and the tangent to each of the following curves:

1000. $r=a\varphi$ (spiral of Archimedes).

1001. $r=\frac{a}{\varphi}$ (hyperbolic spiral).

1002. $r=a\sec^2\frac{\varphi}{2}$ (parabola).

1003. $r=a\cos^2\frac{\varphi}{2}$ (cardioid).

1004. $r=a^{\varphi}$ (logarithmic spiral).

1005. $r^2=a^2\cos 2\varphi$ (lemniscate).

Compute the curvature of the given curves at the indicated points:

1006. $y=x^4-4x^3-18x^2$ at the coordinate origin.

1007. $x^2+xy+y^2=3$ at the point (1, 1).

1008. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the vertices $A(a, 0)$ and $B(0, b)$.

1009. $x=t^2$, $y=t^3$ at the point (1, 1).

1010. $r^2=2a^2\cos 2\varphi$ at the vertices $\varphi=0$ and $\varphi=\pi$.

1011. At what point of the parabola $y^2=8x$ is the curvature equal to 0.128?

1012. Find the vertex of the curve $y=e^x$.

Find the radii of curvature (at any point) of the given lines:

1013. $y=x^3$ (cubic parabola).

1014. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (ellipse).

1015. $x=\frac{y^2}{4}-\frac{\ln y}{2}$.

1016. $x=a\cos^3 t$; $y=a\sin^3 t$ (astroid).

1017. $x=a(\cos t+t\sin t)$; $y=a(\sin t-t\cos t)$ involute of a circle).

1018. $r=ae^{k\varphi}$ (logarithmic spiral).

1019. $r=a(1+\cos\varphi)$ (cardioid).

1020. Find the least value of the radius of curvature of the parabola $y^2=2px$.

1021. Prove that the radius of curvature of the catenary $y=a\cosh\frac{x}{a}$ is equal to a segment of the normal.

Compute the coordinates of the centre of curvature of the given curves at the indicated points:

1022. $xy=1$ at the point $(1, 1)$.
1023. $ay^2=x^3$ at the point (a, a).
Write the equations of the circles of curvature of the given curves at the indicated points:
1024. $y=x^2-6x+10$ at the point $(3, 1)$.
1025. $y=e^x$ at the point $(0, 1)$.
Find the evolutes of the curves:
1026. $y^2=2px$ (parabola).
1027. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (ellipse).
1028. Prove that the evolute of the cycloid

$$x=a(t-\sin t), \quad y=a(1-\cos t)$$

is a displaced cycloid.
1029. Prove that the evolute of the logarithmic spiral

$$r=ae^{k\varphi}$$

is also a logarithmic spiral with the same pole.
1030. Show that the curve (the *involute of a circle*)

$$x=a(\cos t+t\sin t), \quad y=a(\sin t-t\cos t)$$

is the involute of the circle $x=a\cos t$; $y=a\sin t$.

Chapter IV

INDEFINITE INTEGRALS

Sec. 1. Direct Integration

1°. Basic rules of integration.

1) If $F'(x)=f(x)$, then

$$\int f(x)\,dx=F(x)+C,$$

where C is an arbitrary constant.

2) $\int Af(x)\,dx=A\int f(x)\,dx$, where A is a constant quantity.

3) $\int [f_1(x)\pm f_2(x)]\,dx=\int f_1(x)\,dx\pm\int f_2(x)\,dx$.

4) If $\int f(x)\,dx=F(x)+C$ and $u=\varphi(x)$, then

$$\int f(u)\,du=F(u)+C.$$

In particular,

$$\int f(ax+b)\,dx=\frac{1}{a}F(ax+b)+C\quad(a\neq 0).$$

2°. Table of standard integrals.

I. $\int x^n\,dx=\frac{x^{n+1}}{n+1}+C,\quad n\neq -1.$

II. $\int \frac{dx}{x}=\ln|x|+C.$

III. $\int \frac{dx}{x^2+a^2}=\frac{1}{a}\operatorname{arc\,tan}\frac{x}{a}+C=-\frac{1}{a}\operatorname{arc\,cot}\frac{x}{a}+C\quad(a\neq 0).$

IV $\int \frac{dx}{x^2-a^2}=\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C\quad(a\neq 0).$

$\int \frac{dx}{a^2-x^2}=\frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|+C\quad(a\neq 0).$

V. $\int \frac{dx}{\sqrt{x^2+a}}=\ln\left|x+\sqrt{x^2+a}\right|+C\quad(a\neq 0).$

VI. $\int \frac{dx}{\sqrt{a^2-x^2}}=\operatorname{arc\,sin}\frac{x}{a}+C=-\operatorname{arc\,cos}\frac{x}{a}+C\quad(a>0).$

VII. $\int a^x\,dx=\frac{a^x}{\ln a}+C\quad(a>0);\ \int e^x\,dx=e^x+C.$

VIII. $\int \sin x\,dx = -\cos x + C.$

IX. $\int \cos x\,dx = \sin x + C.$

X $\int \frac{dx}{\cos^2 x} = \tan x + C.$

XI. $\int \frac{dx}{\sin^2 x} = -\cot x + C.$

XII $\int \frac{dx}{\sin x} = \ln\left|\tan\frac{x}{2}\right| + C = \ln|\operatorname{cosec} x - \cot x| + C.$

XIII. $\int \frac{dx}{\cos x} = \ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + C = \ln|\tan x + \sec x| + C.$

XIV. $\int \sinh x\,dx = \cosh x + C.$

XV. $\int \cosh x\,dx = \sinh x + C.$

XVI. $\int \frac{dx}{\cosh^2 x} = \tanh x + C.$

XVII. $\int \frac{dx}{\sinh^2 x} = -\coth x + C.$

Example 1.

$$\int (ax^2+bx+c)\,dx = \int ax^2dx + \int bx\,dx + \int c\,dx =$$
$$= a\int x^2dx + b\int x\,dx + c\int dx = a\frac{x^3}{3} + b\frac{x^2}{2} + cx + C.$$

Applying the basic rules 1, 2, 3 and the formulas of integration, find the following integrals:

1031. $\int 5a^2x^6dx.$

1032. $\int (6x^2+8x+3)\,dx.$

1033. $\int x(x+a)(x+b)\,dx.$

1034. $\int (a+bx^3)^2\,dx.$

1035. $\int \sqrt{2px}\,dx.$

1036. $\int \frac{dx}{\sqrt[n]{x}}.$

1037. $\int (nx)^{\frac{1-n}{n}}\,dx.$

1038. $\int \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^3 dx.$

1039. $\int (\sqrt{x}+1)(x-\sqrt{x}+1)\,dx.$

1040. $\int \frac{(x^2+1)(x^2-2)}{\sqrt[3]{x^2}}\,dx.$

1041. $\int \frac{(x^m-x^n)^2}{\sqrt{x}}\,dx.$

1042. $\int \frac{(\sqrt{a}-\sqrt{x})^4}{\sqrt{ax}}\,dx.$

1043. $\int \frac{dx}{x^2+7}.$

1044. $\int \frac{dx}{x^2-10}.$

1045. $\int \frac{dx}{\sqrt{4+x^2}}.$

1046. $\int \frac{dx}{\sqrt{8-x^2}}.$

1047. $\int \frac{\sqrt{2+x^2}-\sqrt{2-x^2}}{\sqrt{4-x^4}}\,dx.$

1048*. a) $\int \tan^2 x\,dx$;
b) $\int \tanh^2 x\,dx$.

1049. a) $\int \cot^2 x\,dx$;
b) $\int \coth^2 x\,dx$.

1050. $\int 3^x e^x dx$.

3°. Integration under the sign of the differential. Rule 4 considerably expands the table of standard integrals: by virtue of this rule the table of integrals holds true irrespective of whether the variable of integration is an independent variable or a differentiable function.

Example 2.

$$\int \frac{dx}{\sqrt{5x-2}}=\frac{1}{5}\int (5x-2)^{-\frac{1}{2}}\,d(5x-2)=$$

$$=\frac{1}{5}\int u^{-\frac{1}{2}}\,du=\frac{1}{5}\cdot\frac{u^{\frac{1}{2}}}{\frac{1}{2}}+C=\frac{2}{5}\,\frac{(5x-2)^{\frac{1}{2}}}{\frac{1}{2}}+C=\frac{2}{5}\sqrt{5x-2}+C,$$

where we put $u=5x-2$. We took advantage of Rule 4 and tabular integral I.

Example 3. $\int \frac{x\,dx}{\sqrt{1+x^4}}=\frac{1}{2}\int \frac{d(x^2)}{\sqrt{1+(x^2)^2}}=\frac{1}{2}\ln(x^2+\sqrt{1+x^4})+C.$

We implied $u=x^2$, and use was made of Rule 4 and tabular integral V.

Example 4. $\int x^2 e^{x^3}\,dx=\frac{1}{3}\int e^{x^3}\,d(x^3)=\frac{1}{3}e^{x^3}+C$ by virtue of Rule 4 and tabular integral VII.

In examples 2, 3, and 4 we reduced the given integral to the following form before making use of a tabular integral:

$$\int f(\varphi(x))\,\varphi'(x)\,dx=\int f(u)\,du, \text{ where } u=\varphi(x).$$

This type of transformation is called *integration under the differential sign.*

Some common transformations of differentials, which were used in Examples 2 and 3, are:

a) $dx=\frac{1}{a}\,d(ax+b)\quad(a\neq 0)$; b) $x\,dx=\frac{1}{2}\,d(x^2)$ and so on.

Using the basic rules and formulas of integration, find the following integrals:

1051.** $\int \frac{a\,dx}{a-x}$.

1052.** $\int \frac{2x+3}{2x+1}\,dx$.

1053. $\int \frac{1-3x}{3+2x}\,dx$.

1054. $\int \frac{x\,dx}{a+bx}$.

1055. $\int \frac{ax+b}{\alpha x+\beta}\,dx$.

1056. $\int \frac{x^2+1}{x-1}\,dx$.

1057. $\int \frac{x^2+5x+7}{x+3}\,dx$.

1058. $\int \frac{x^4+x^2+1}{x-1}\,dx$.

1059. $\int \left(a+\frac{b}{x-a}\right)^2 dx.$

1060*. $\int \frac{x}{(x+1)^2}\,dx.$

1061. $\int \frac{b\,dy}{\sqrt{1-y}}.$

1062. $\int \sqrt{a-bx}\,dx.$

1063*. $\int \frac{x}{\sqrt{x^2+1}}\,dx.$

1064. $\int \frac{\sqrt{x}+\ln x}{x}\,dx.$

1065. $\int \frac{dx}{3x^2+5}.$

1066. $\int \frac{dx}{7x^2-8}.$

1067. $\int \frac{dx}{(a+b)-(a-b)\,x^2}$ $(0<b<a).$

1068. $\int \frac{x^2}{x^2+2}\,dx.$

1069. $\int \frac{x^3}{a^2-x^2}\,dx.$

1070. $\int \frac{x^2-5x+6}{x^2+4}\,dx.$

1071. $\int \frac{dx}{\sqrt{7+8x^2}}.$

1072. $\int \frac{dx}{\sqrt{7-5x^2}}.$

1073. $\int \frac{2x-5}{3x^2-2}\,dx.$

1074. $\int \frac{3-2x}{5x^2+7}\,dx.$

1075. $\int \frac{3x+1}{\sqrt{5x^2+1}}\,dx.$

1076. $\int \frac{x+3}{\sqrt{x^2-4}}\,dx.$

1077. $\int \frac{x\,dx}{x^2-5}.$

1078. $\int \frac{x\,dx}{2x^2+3}.$

1079. $\int \frac{ax+b}{a^2x^2+b^2}\,dx.$

1080. $\int \frac{x\,dx}{\sqrt{a^4-x^4}}.$

1081. $\int \frac{x^2}{1+x^6}\,dx.$

1082. $\int \frac{x^2\,dx}{\sqrt{x^6-1}}.$

1083. $\int \sqrt{\frac{\arcsin x}{1-x^2}}\,dx.$

1084. $\int \frac{\arctan \frac{x}{2}}{4+x^2}\,dx.$

1085. $\int \frac{x-\sqrt{\arctan 2x}}{1+4x^2}\,dx.$

1086. $\int \frac{dx}{\sqrt{(1+x^2)\ln\left(x+\sqrt{1+x^2}\right)}}.$

1087. $\int ae^{-mx}\,dx.$

1088. $\int 4^{2-3x}\,dx.$

1089. $\int (e^t-e^{-t})\,dt.$

1090. $\int \left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)^2 dx.$

1091. $\int \frac{(a^x-b^x)^2}{a^xb^x}\,dx.$

1092. $\int \frac{a^{2x}-1}{\sqrt{a^x}}\,dx.$

1093. $\int e^{-(x^2+1)}\,x\,dx.$

1094. $\int x\cdot 7^{x^2}\,dx.$

1095. $\int \frac{e^{\frac{1}{x}}}{x^2}\,dx.$

1096. $\int 5^{\sqrt{x}}\,\frac{dx}{\sqrt{x}}.$

1097. $\int \frac{e^x}{e^x-1}\,dx.$

1098. $\int e^x \sqrt{a-be^x}\,dx.$

1099. $\int \left(e^{\frac{x}{a}}+1\right)^{\frac{1}{3}} e^{\frac{x}{a}}\,dx.$

1100*. $\int \frac{dx}{2^x+3}.$

1101. $\int \frac{a^x\,dx}{1+a^{2x}}.$

1102. $\int \frac{e^{-bx}}{1-e^{-2bx}}\,dx.$

1103. $\int \frac{e^t\,dt}{\sqrt{1-e^{2t}}}.$

1104. $\int \sin(a+bx)\,dx.$

1105. $\int \cos\frac{x}{\sqrt{2}}\,dx.$

1106. $\int (\cos ax+\sin ax)^2\,dx.$

1107. $\int \cos\sqrt{x}\,\frac{dx}{\sqrt{x}}.$

1108. $\int \sin(\lg x)\frac{dx}{x}.$

1109*. $\int \sin^2 x\,dx.$

1110*. $\int \cos^2 x\,dx.$

1111. $\int \sec^2(ax+b)\,dx.$

1112. $\int \cot^2 ax\,dx.$

1113. $\int \frac{dx}{\sin\frac{x}{a}}.$

1114. $\int \frac{dx}{3\cos\left(5x-\frac{\pi}{4}\right)}.$

1115. $\int \frac{dx}{\sin(ax+b)}.$

1116. $\int \frac{x\,dx}{\cos^2 x^2}.$

1117. $\int x\sin(1-x^2)\,dx.$

1118. $\int \left(\frac{1}{\sin x\sqrt{2}}-1\right)^2 dx.$

1119. $\int \tan x\,dx.$

1120. $\int \cot x\,dx.$

1121. $\int \cot\frac{x}{a-b}\,dx.$

1122. $\int \frac{dx}{\tan\frac{x}{5}}.$

1123. $\int \tan\sqrt{x}\,\frac{dx}{\sqrt{x}}.$

1124. $\int x\cot(x^2+1)\,dx.$

1125. $\int \frac{dx}{\sin x\cos x}.$

1126. $\int \cos\frac{x}{a}\sin\frac{x}{a}\,dx.$

1127. $\int \sin^3 6x\cos 6x\,dx.$

1128. $\int \frac{\cos ax}{\sin^5 ax}\,dx.$

1129. $\int \frac{\sin 3x}{3+\cos 3x}\,dx.$

1130. $\int \frac{\sin x\cos x}{\sqrt{\cos^2 x-\sin^2 x}}\,dx.$

1131. $\int \sqrt{1+3\cos^2 x}\,\sin 2x\,dx.$

1132. $\int \tan^3\frac{x}{3}\sec^2\frac{x}{3}\,dx.$

1133. $\int \frac{\sqrt{\tan x}}{\cos^2 x}\,dx.$

1134. $\int \frac{\cot^{\frac{2}{3}} x}{\sin^2 x}\,dx.$

1135. $\int \frac{1+\sin 3x}{\cos^2 3x}\,dx.$

1136. $\int \frac{(\cos ax+\sin ax)^2}{\sin ax}\,dx.$

1137. $\int \frac{\operatorname{cosec}^2 3x}{b-a\cot 3x}\,dx.$

1138. $\int (2\sinh 5x-3\cosh 5x)\,dx.$

1139. $\int \sinh^2 x\,dx.$

1140. $\int \frac{dx}{\sinh x}$.

1141. $\int \frac{dx}{\cosh x}$.

1142. $\int \frac{dx}{\sinh x \cosh x}$.

1143. $\int \tanh x\, dx$.

1144. $\int \coth x\, dx$.

Find the indefinite integrals:

1145. $\int x \sqrt[5]{5-x^2}\, dx$.

1146. $\int \frac{x^3-1}{x^4-4x+1}\, dx$.

1147. $\int \frac{x^3}{x^8+5}\, dx$.

1148. $\int xe^{-x^2}\, dx$.

1149. $\int \frac{3-\sqrt{2+3x^2}}{2+3x^2}\, dx$.

1150. $\int \frac{x^3-1}{x-1}\, dx$.

1151. $\int \frac{dx}{\sqrt{e^x}}$.

1152. $\int \frac{1-\sin x}{x+\cos x}\, dx$.

1153. $\int \frac{\tan 3x - \cot 3x}{\sin 3x}\, dx$.

1154. $\int \frac{dx}{x \ln^2 x}$.

1155. $\int \frac{\sec^2 x}{\sqrt{\tan^2 x - 2}}\, dx$.

1156. $\int \left(2+\frac{x}{2x^2+1}\right) \frac{dx}{2x^2+1}$.

1157. $\int a^{\sin x} \cos x\, dx$.

1158. $\int \frac{x^2}{\sqrt[3]{x^3+1}}\, dx$.

1159. $\int \frac{x\, dx}{\sqrt{1-x^4}}$.

1160. $\int \tan^2 ax\, dx$.

1161. $\int \sin^2 \frac{x}{2}\, dx$.

1162. $\int \frac{\sec^2 x\, dx}{\sqrt{4-\tan^2 x}}$.

1163. $\int \frac{dx}{\cos \frac{x}{a}}$.

1164. $\int \frac{\sqrt[3]{1+\ln x}}{x}\, dx$.

1165. $\int \tan \sqrt{x-1}\, \frac{dx}{\sqrt{x-1}}$.

1166. $\int \frac{x\, dx}{\sin x^2}$.

1167. $\int \frac{e^{\arctan x} + x \ln(1+x^2)+1}{1+x^2}\, dx$.

1168. $\int \frac{\sin x - \cos x}{\sin x + \cos x}\, dx$.

1169. $\int \frac{\left(1-\sin \frac{x}{\sqrt{2}}\right)^2}{\sin \frac{x}{\sqrt{2}}}\, dx$.

1170. $\int \frac{x^2}{x^2-2}\, dx$.

1171. $\int \frac{(1+x)^2}{x(1+x^2)}\, dx$.

1172. $\int e^{\sin^2 x} \sin 2x\, dx$.

1173. $\int \frac{5-3x}{\sqrt{4-3x^2}}\, dx$.

1174. $\int \frac{dx}{e^x+1}$.

1175. $\int \frac{dx}{(a+b)+(a-b)x^2}$ $(0 < b < a)$.

1176. $\int \frac{e^x}{\sqrt{e^{2x}-2}}\, dx$.

1177. $\int \frac{dx}{\sin ax \cos ax}$.

1178. $\int \sin\left(\frac{2\pi t}{T}+\varphi_0\right) dt.$

1179. $\int \frac{dx}{x(4-\ln^2 x)}.$

1180. $\int \frac{\arccos\frac{x}{2}}{\sqrt{4-x^2}} dx.$

1181. $\int e^{-\tan x}\sec^2 x\, dx.$

1182. $\int \frac{\sin x\cos x}{\sqrt{2-\sin^4 x}} dx.$

1183. $\int \frac{dx}{\sin^2 x\cos^2 x}.$

1184. $\int \frac{\arcsin x+x}{\sqrt{1-x^2}} dx.$

1185. $\int \frac{\sec x\tan x}{\sqrt{\sec^2 x+1}} dx.$

1186. $\int \frac{\cos 2x}{4+\cos^2 2x} dx.$

1187. $\int \frac{dx}{1+\cos^2 x}.$

1188. $\int \sqrt{\frac{\ln(x+\sqrt{x^2+1})}{1+x^2}}\, dx.$

1189. $\int x^2\cos(x^3+3)\, dx.$

1190. $\int \frac{3^{\tanh x}}{\cosh^2 x} dx.$

Sec. 2. Integration by Substitution

1°. Change of variable in an indefinite integral. Putting

$$x=\varphi(t),$$

where t is a new variable and φ is a continuously differentiable function, we will have:

$$\int f(x)\,dx=\int f[\varphi(t)]\,\varphi'(t)\,dt. \qquad (1)$$

The attempt is made to choose the function φ in such a way that the right side of (1) becomes more convenient for integration.

Example 1. Find

$$\int x\sqrt{x-1}\,dx.$$

Solution. It is natural to put $t=\sqrt{x-1}$, whence $x=t^2+1$ and $dx=2t\,dt$. Hence,

$$\int x\sqrt{x-1}\,dx=\int (t^2+1)\,t\cdot 2t\,dt=2\int (t^4+t^2)\,dt=$$

$$=\frac{2}{5}t^5+\frac{2}{3}t^3+C=\frac{2}{5}(x-1)^{\frac{5}{2}}+\frac{2}{3}(x-1)^{\frac{3}{2}}+C.$$

Sometimes substitutions of the form

$$u=\varphi(x)$$

are used.

Suppose we succeeded in transforming the integrand $f(x)\,dx$ to the form

$$f(x)\,dx=g(u)\,du, \text{ where } u=\varphi(x).$$

If $\int g(u)\,du$ is known, that is,

$$\int g(u)\,du = F(u) + C,$$

then

$$\int f(x)\,dx = F[\varphi(x)] + C.$$

Actually, we have already made use of this method in Sec. 1,3°.

Examples 2, 3, 4 (Sec. 1) may be solved as follows:

Example 2. $u = 5x-2$; $du = 5dx$; $dx = \frac{1}{5}\,du$.

$$\int \frac{dx}{\sqrt{5x-2}} = \frac{1}{5}\frac{du}{\sqrt{u}} = \frac{1}{5}\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C = \frac{2}{5}\sqrt{5x-2} + C.$$

Example 3. $u = x^2$; $du = 2x\,dx$; $x\,dx = \frac{du}{2}$.

$$\int \frac{x\,dx}{\sqrt{1+x^4}} = \frac{1}{2}\int \frac{du}{\sqrt{1+u^2}} = \frac{1}{2}\ln(u+\sqrt{1+u^2}) + C = \frac{1}{2}\ln(x^2+\sqrt{1+x^4}) + C.$$

Example 4. $u = x^3$; $du = 3x^2\,dx$; $x^2\,dx = \frac{du}{3}$.

$$\int x^2 e^{x^3}\,dx = \frac{1}{3}\int e^u\,du = \frac{1}{3}e^u + C = \frac{1}{3}e^{x^3} + C.$$

2°. Trigonometric substitutions.

1) If an integral contains the radical $\sqrt{a^2-x^2}$, the usual thing is to put $x = a\sin t$; whence

$$\sqrt{a^2-x^2} = a\cos t.$$

2) If an integral contains the radical $\sqrt{x^2-a^2}$, we put $x = a\sec t$, whence

$$\sqrt{x^2-a^2} = a\tan t.$$

3) If an integral contains the radical $\sqrt{x^2+a^2}$, we put $x = a\tan t$; whence

$$\sqrt{x^2+a^2} = a\sec t.$$

It should be noted that trigonometric substitutions do not always turn out to be advantageous.

It is sometimes more convenient to make use of *hyperbolic substitutions*, which are similar to trigonometric substitutions (see Example 1209).

For more details about trigonometric and hyperbolic substitutions, see Sec. 9.

Example 5. Find

$$\int \frac{\sqrt{x^2+1}}{x^2}\,dx.$$

Solution. Put $x=\tan t$. Therefore, $dx=\frac{dt}{\cos^2 t}$.

$$\int \frac{\sqrt{x^2+1}}{x^2}\,dx=\int \frac{\sqrt{\tan^2 t+1}}{\tan^2 t}\,\frac{dt}{\cos^2 t}=\int \frac{\sec t\cos^2 t}{\sin^2 t}\,\frac{dt}{\cos^2 t}=$$

$$=\int \frac{dt}{\sin^2 t\cos t}=\int \frac{\sin^2 t+\cos^2 t}{\sin^2 t\cdot\cos t}\,dt=\int \frac{dt}{\cos t}+\int \frac{\cos t}{\sin^2 t}\,dt=$$

$$=\ln|\tan t+\sec t|-\frac{1}{\sin t}+C=\ln|\tan t+\sqrt{1+\tan^2 t}|-$$

$$-\frac{\sqrt{1+\tan^2 t}}{\tan t}+C=\ln|x+\sqrt{x^2+1}|-\frac{\sqrt{x^2+1}}{x}+C.$$

1191. Applying the indicated substitutions, find the following integrals:

a) $\int \frac{dx}{x\sqrt{x^2-2}}$, $\quad x=\frac{1}{t}$;

b) $\int \frac{dx}{e^x+1}$, $\quad x=-\ln t$,

c) $\int x(5x^2-3)^7\,dx$, $\quad 5x^2-3=t$;

d) $\int \frac{x\,dx}{\sqrt{x+1}}$, $\quad t=\sqrt{x+1}$;

e) $\int \frac{\cos x\,dx}{\sqrt{1+\sin^2 x}}$, $\quad t=\sin x$.

Applying suitable substitutions, find the following integrals:

1192. $\int x(2x+5)^{10}\,dx$.

1193. $\int \frac{1+x}{1+\sqrt{x}}\,dx$.

1194. $\int \frac{dx}{x\sqrt{2x+1}}$.

1195. $\int \frac{dx}{\sqrt{e^x-1}}$.

1196. $\int \frac{\ln 2x\,dx}{\ln 4x\,x}$.

1197. $\int \frac{(\arcsin x)^2}{\sqrt{1-x^2}}\,dx$.

1198. $\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx$.

1199. $\int \frac{\sin^3 x}{\sqrt{\cos x}}\,dx$.

1200*. $\int \frac{dx}{x\sqrt{1+x^2}}$.

Applying trigonometric substitutions, find the following integrals:

1201. $\int \frac{x^2\,dx}{\sqrt{1-x^2}}$.

1202. $\int \frac{x^3\,dx}{\sqrt{2-x^2}}$.

1203. $\int \frac{\sqrt{x^2-a^2}}{x}\,dx$.

1204*. $\int \frac{dx}{x\sqrt{x^2-1}}$.

1205. $\int \frac{\sqrt{x^2+1}}{x}\,dx.$ **1206*.** $\int \frac{dx}{x^2\sqrt{4-x^2}}.$

1207. $\int \sqrt{1-x^2}\,dx.$

1208. Evaluate the integral

$$\int \frac{dx}{\sqrt{x(1-x)}}$$

by means of the substitution $x=\sin^2 t$.

1209. Find

$$\int \sqrt{a^2+x^2}\,dx,$$

by applying the hyperbolic substitution $x=a\sinh t$.

Solution. We have: $\sqrt{a^2+x^2}=\sqrt{a^2+a^2\sinh^2 t}=a\cosh t$ and $dx=a\cosh t\,dt$. Whence

$$\int \sqrt{a^2+x^2}\,dx=\int a\cosh t\cdot a\cosh t\,dt=$$
$$=a^2\int \cosh^2 t\,dt=a^2\int \frac{\cosh 2t+1}{2}\,dt=\frac{a^2}{2}\left(\frac{1}{2}\sinh 2t+t\right)+C=$$
$$=\frac{a^2}{2}(\sinh t\cosh t+t)+C.$$

Since

$$\sinh t=\frac{x}{a},\quad \cosh t=\frac{\sqrt{a^2+x^2}}{a}$$

and

$$e^t=\cosh t+\sinh t=\frac{x+\sqrt{a^2+x^2}}{a}$$

we finally get

$$\int \sqrt{a^2+x^2}\,dx=\frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\ln(x+\sqrt{a^2+x^2})+C_1,$$

where $C_1=C-\frac{a^2}{2}\ln a$ is a new arbitrary constant.

1210. Find

$$\int \frac{x^2\,dx}{\sqrt{x^2-a^2}},$$

putting $x=a\cosh t$.

Sec. 3. Integration by Parts

A formula for integration by parts. If $u=\varphi(x)$ and $v=\psi(x)$ are differentiable functions, then

$$\int u\,dv=uv-\int v\,du.$$

Example 1. Find

$$\int x \ln x \, dx.$$

Putting $u = \ln x$, $dv = x\,dx$, we have $du = \frac{dx}{x}$, $v = \frac{x^2}{2}$ Whence

$$\int x \ln x \, dx = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \frac{dx}{x} = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C.$$

Sometimes, to reduce a given integral to tabular form, one has to apply the formula of integration by parts several times. In certain cases, integration by parts yields an equation from which the desired integral is determined.

Example 2. Find

$$\int e^x \cos x \, dx.$$

We have

$$\int e^x \cos x \, dx = \int e^x d(\sin x) = e^x \sin x - \int e^x \sin x \, dx = e^x \sin x +$$
$$+ \int e^x \, d(\cos x) = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx.$$

Hence,

$$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx,$$

whence

$$\int e^x \cos x \, dx = \frac{e^x}{2} (\sin x + \cos x) + C.$$

Applying the formula of integration by parts, find the following integrals:

1211. $\int \ln x \, dx.$

1212. $\int \arctan x \, dx.$

1213. $\int \arcsin x \, dx.$

1214. $\int x \sin x \, dx.$

1215. $\int x \cos 3x \, dx.$

1216. $\int \frac{x}{e^x} \, dx.$

1217. $\int x \cdot 2^{-x} \, dx.$

1218.** $\int x^2 e^{3x} \, dx.$

1219*. $\int (x^2 - 2x + 5) e^{-x} \, dx.$

1220*. $\int x^3 e^{-\frac{x}{3}} \, dx.$

1221. $\int x \sin x \cos x \, dx$

1222* $\int (x^2 + 5x + 6) \cos 2x \, dx.$

1223. $\int x^2 \ln x \, dx.$

1224. $\int \ln^2 x \, dx.$

1225. $\int \frac{\ln x}{x^3} \, dx.$

1226. $\int \frac{\ln x}{\sqrt{x}} \, dx.$

1227. $\int x \arctan x \, dx.$

1228. $\int x \arcsin x \, dx.$

1229. $\int \ln (x + \sqrt{1 + x^2}) \, dx.$

1230. $\int \frac{x \, dx}{\sin^2 x}.$

1231. $\int \frac{x\cos x}{\sin^2 x}\,dx.$

1232. $\int e^x \sin x\,dx.$

1233. $\int 3^x \cos x\,dx.$

1234. $\int e^{ax} \sin bx\,dx.$

1235. $\int \sin(\ln x)\,dx.$

Applying various methods, find the following integrals:

1236. $\int x^3 e^{-x^2}\,dx.$

1237. $\int e^{\sqrt{x}}\,dx.$

1238. $\int (x^2-2x+3)\ln x\,dx.$

1239. $\int x\ln\frac{1-x}{1+x}\,dx.$

1240. $\int \frac{\ln^2 x}{x^2}\,dx.$

1241. $\int \frac{\ln(\ln x)}{x}\,dx.$

1242. $\int x^2 \arctan 3x\,dx.$

1243. $\int x(\arctan x)^2\,dx.$

1244. $\int (\arcsin x)^2\,dx.$

1245. $\int \frac{\arcsin x}{x^2}\,dx.$

1246. $\int \frac{\arcsin\sqrt{x}}{\sqrt{1-x}}\,dx.$

1247. $\int x\tan^2 2x\,dx.$

1248. $\int \frac{\sin^2 x}{e^x}\,dx.$

1249. $\int \cos^2(\ln x)\,dx.$

1250**. $\int \frac{x^2}{(x^2+1)^2}\,dx.$

1251*. $\int \frac{dx}{(x^2+a^2)^2}.$

1252*. $\int \sqrt{a^2-x^2}\,dx.$

1253*. $\int \sqrt{A+x^2}\,dx.$

1254*. $\int \frac{x^2\,dx}{\sqrt{9-x^2}}.$

Sec. 4. Standard Integrals Containing a Quadratic Trinomial

1°. Integrals of the form

$$\int \frac{mx+n}{ax^2+bx+c}\,dx.$$

The principal calculation procedure is to reduce the quadratic trinomial to the form

$$ax^2+bx+c=a(x+k)^2+l, \qquad (1)$$

where k and l are constants. To perform the transformations in (1), it is best to take the perfect square out of the quadratic trinomial. The following substitution may also be used:

$$2ax+b=t.$$

If $m=0$, then, reducing the quadratic trinomial to the form (1), we get the tabular integrals III or IV (see Table).

Example 1.

$$\int \frac{dx}{2x^2-5x+7}=\frac{1}{2}\int \frac{dx}{\left(x^2-2\cdot\frac{5}{4}x+\frac{25}{16}\right)+\left(\frac{7}{2}-\frac{25}{16}\right)}=$$

$$=\frac{1}{2}\int \frac{d\left(x-\frac{5}{4}\right)}{\left(x-\frac{5}{4}\right)^2+\frac{31}{16}}=\frac{1}{2}\,\frac{1}{\frac{\sqrt{31}}{4}}\operatorname{arc\,tan}\frac{x-\frac{5}{4}}{\frac{\sqrt{31}}{4}}+C=$$

$$=\frac{2}{\sqrt{31}}\operatorname{arc\,tan}\frac{4x-5}{\sqrt{31}}+C.$$

If $m\neq 0$, then from the numerator we can take the derivative $2ax+b$ out of the quadratic trinomial

$$\int \frac{mx+n}{ax^2+bx+c}\,dx=\int \frac{\frac{m}{2a}(2ax+b)+\left(n-\frac{mb}{2a}\right)}{ax^2+bx+c}\,dx=$$

$$=\frac{m}{2a}\ln|ax^2+bx+c|+\left(n-\frac{mb}{2a}\right)\int\frac{dx}{ax^2+bx+c},$$

and thus we arrive at the integral discussed above.

Example 2.

$$\int \frac{x-1}{x^2-x-1}\,dx=\int \frac{\frac{1}{2}(2x-1)-\frac{1}{2}}{x^2-x-1}\,dx=\frac{1}{2}\ln|x^2-x-1|-$$

$$-\frac{1}{2}\int \frac{d\left(x-\frac{1}{2}\right)}{\left(x-\frac{1}{2}\right)^2-\frac{5}{4}}=\frac{1}{2}\ln|x^2-x-1|-\frac{1}{2\sqrt{5}}\ln\left|\frac{2x-1-\sqrt{5}}{2x-1+\sqrt{5}}\right|+C.$$

2°. **Integrals of the form** $\int \frac{mx+n}{\sqrt{ax^2+bx+c}}\,dx$. The methods of calculation are similar to those analyzed above. The integral is finally reduced to tabular integral V, if $a>0$, and VI, if $a<0$.

Example 3.

$$\int \frac{dx}{\sqrt{2+3x-2x^2}}=\frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\frac{25}{16}-\left(x-\frac{3}{4}\right)^2}}=\frac{1}{\sqrt{2}}\operatorname{arc\,sin}\frac{4x-3}{5}+C.$$

Example 4.

$$\int \frac{x+3}{\sqrt{x^2+2x+2}}\,dx=\frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+2}}\,dx+2\int \frac{dx}{\sqrt{(x+1)^2+1}}=$$

$$=\sqrt{x^2+2x+2}+2\ln\left(x+1+\sqrt{x^2+2x+2}\right)+C.$$

3°. Integrals of the form $\int \frac{dx}{(mx+n)\sqrt{ax^2+bx+c}}$. By means of the inverse substitution

$$\frac{1}{mx+n}=t$$

these integrals are reduced to integrals of the form 2°.

Example 5. Find

$$\int \frac{dx}{(x+1)\sqrt{x^2+1}}.$$

Solution. We put

$$x+1=\frac{1}{t},$$

whence

$$dx=-\frac{dt}{t^2}.$$

We have:

$$\int \frac{dx}{(x+1)\sqrt{x^2+1}}=\int \frac{-\frac{dt}{t^2}}{\frac{1}{t}\sqrt{\left(\frac{1}{t}-1\right)^2+1}}=-\int \frac{dt}{\sqrt{1-2t+2t^2}}=$$

$$=-\frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{\left(t-\frac{1}{2}\right)^2+\frac{1}{4}}}=-\frac{1}{\sqrt{2}}\ln\left|t-\frac{1}{2}+\sqrt{t^2-t+\frac{1}{2}}\right|+$$

$$+C=-\frac{1}{\sqrt{2}}\ln\left|\frac{1-x+\sqrt{2(x^2+1)}}{x+1}\right|+C.$$

4°. Integrals of the form $\int \sqrt{ax^2+bx+c}\,dx$. By taking the perfect square out of the quadratic trinomial, the given integral is reduced to one of the following two basic integrals (see examples 1252 and 1253):

1) $\int \sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\frac{x}{a}+C;$

$(a>0);$

2) $\int \sqrt{x^2+A}\,dx=\frac{x}{2}\sqrt{x^2+A}+\frac{A}{2}\ln\left|x+\sqrt{x^2+A}\right|+C.$

Example 6.

$$\int \sqrt{1-2x-x^2}\,dx=\int \sqrt{2-(1+x)^2}\,d(1+x)=$$

$$=\frac{1+x}{2}\sqrt{1-2x-x^2}+\arcsin\frac{1+x}{\sqrt{2}}+C.$$

Find the following integrals:

1255. $\int \frac{dx}{x^2+2x+5}$.

1256. $\int \frac{dx}{x^2+2x}$.

1257. $\int \frac{dx}{3x^2-x+1}$.

1258. $\int \frac{x\,dx}{x^2-7x+13}$.

1259. $\int \frac{3x-2}{x^2-4x+5}\,dx$.

1260. $\int \frac{(x-1)^2}{x^2+3x+4}\,dx$.

1261. $\int \frac{x^2\,dx}{x^2-6x+10}$

1262. $\int \frac{dx}{\sqrt{2+3x-2x^2}}$.

1263. $\int \frac{dx}{\sqrt{x-x^2}}$

1264. $\int \frac{dx}{\sqrt{x^2+px+q}}$.

1265. $\int \frac{3x-6}{\sqrt{x^2-4x+5}}\,dx$.

1266. $\int \frac{2x-8}{\sqrt{1-x-x^2}}\,dx$.

1267. $\int \frac{x}{\sqrt{5x^2-2x+1}}\,dx$.

1268. $\int \frac{dx}{x\sqrt{1-x^2}}$.

1269. $\int \frac{dx}{x\sqrt{x^2+x-1}}$.

1270. $\int \frac{dx}{(x-1)\sqrt{x^2-2}}$.

1271. $\int \frac{dx}{(x+1)\sqrt{x^2+2x}}$.

1272. $\int \sqrt{x^2+2x+5}\,dx$.

1273. $\int \sqrt{x-x^2}\,dx$

1274. $\int \sqrt{2-x-x^2}\,dx$.

1275. $\int \frac{x\,dx}{x^4-4x^2+3}$.

1276. $\int \frac{\cos x}{\sin^2 x-6\sin x+12}\,dx$.

1277. $\int \frac{e^x dx}{\sqrt{1+e^x+e^{2x}}}$

1278. $\int \frac{\sin x\,dx}{\sqrt{\cos^2 x+4\cos x+1}}$.

1279. $\int \frac{\ln x\,dx}{x\sqrt{1-4\ln x-\ln^2 x}}$.

Sec. 5. Integration of Rational Functions

1°. **The method of undetermined coefficients.** Integration of a rational function, after taking out the whole part, reduces to integration of the *proper rational fraction*

$$\frac{P(x)}{Q(x)}, \tag{1}$$

where $P(x)$ and $Q(x)$ are integral polynomials, and the degree of the numerator $P(x)$ is lower than that of the denominator $Q(x)$.

If

$$Q(x)=(x-a)^{\alpha}\ldots(x-l)^{\lambda},$$

where $a, \ldots, l$ are real distinct roots of the polynomial $Q(x)$, and $\alpha, \ldots, \lambda$ are natural numbers (root multiplicities), then decomposition of (1) into partial fractions is justified:

$$\frac{P(x)}{Q(x)}=\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\ldots+\frac{A_\alpha}{(x-a)^\alpha}+\ldots$$

$$\ldots+\frac{L_1}{x-l}+\frac{L_2}{(x-l)^2}+\ldots+\frac{L_\lambda}{(x-l)^\lambda}. \tag{2}$$

To calculate the undetermined coefficients $A_1, A_2, \ldots$, both sides of the identity (2) are reduced to an integral form, and then the coefficients of like powers of the variable x are equated (**first method**). These coefficients may likewise be determined by putting [in equation (2) or in an equivalent equation] x equal to suitably chosen numbers (**second method**).

Example 1. Find

$$\int \frac{x\,dx}{(x-1)(x+1)^2}=I.$$

Solution. We have:

$$\frac{x}{(x-1)(x+1)^2}=\frac{A}{x-1}+\frac{B_1}{x+1}+\frac{B_2}{(x+1)^2}.$$

Whence

$$x\equiv A(x+1)^2+B_1(x-1)(x+1)+B_2(x-1). \tag{3}$$

a) *First method of determining the coefficients.* We rewrite identity (3) in the form $x\equiv(A+B_1)x^2+(2A+B_2)x+(A-B_1-B_2)$ Equating the coefficients of identical powers of x, we get:

$$0=A+B_1;\quad 1=2A+B_2;\quad 0=A-B_1-B_2.$$

Whence

$$A=\frac{1}{4};\quad B_1=-\frac{1}{4};\quad B_2=\frac{1}{2}.$$

b) *Second method of determining the coefficients.* Putting $x=1$ in identity (3), we will have:

$$1=A\cdot 4,\quad \text{i. e.,}\quad A=1/4.$$

Putting $x=-1$, we get:

$$-1=-B_2\cdot 2,\quad \text{i. e.,}\quad B_2=1/2.$$

Further, putting $x=0$, we will have:

$$0=A-B_1-B_2,$$

or $B_1=A-B_2=-1/4$.

Hence,

$$I=\frac{1}{4}\int\frac{dx}{x-1}-\frac{1}{4}\int\frac{dx}{x+1}+\frac{1}{2}\int\frac{dx}{(x+1)^2}=$$

$$=\frac{1}{4}\ln|x-1|-\frac{1}{4}\ln|x+1|-\frac{1}{2(x+1)}+C=$$

$$=-\frac{1}{2(x+1)}+\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right|+C.$$

Example 2. Find

$$\int\frac{dx}{x^3-2x^2+x}=I.$$

Solution. We have:

$$\frac{1}{x^3-2x^2+x}=\frac{1}{x(x-1)^2}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

and

$$1=A(x-1)^2+Bx(x-1)+Cx. \tag{4}$$

When solving this example it is advisable to combine the two methods of determining coefficients. Applying the second method, we put $x=0$ in identity (4). We get $1=A$. Then, putting $x=1$, we get $1=C$. Further, applying the first method, we equate the coefficients of x^2 in identity (4), and get:

$$0=A+B,\quad \text{i. e.,}\quad B=-1.$$

Hence,

$$A=1,\quad B=-1,\quad \text{and}\quad C=1.$$

Consequently,

$$I=\int \frac{dx}{x}-\int \frac{dx}{x-1}+\int \frac{dx}{(x-1)^2}=\ln|x|-\ln|x-1|-\frac{1}{x-1}+C.$$

If the polynomial $Q(x)$ has complex roots $a \pm ib$ of multiplicity k, then partial fractions of the form

$$\frac{A_1x+B_1}{x^2+px+q}+\ldots+\frac{A_kx+B_k}{(x^2+px+q)^k} \qquad (5)$$

will enter into the expansion (2). Here,

$$x^2+px+q=[x-(a+ib)]\,[x-(a-ib)]$$

and A_1, B_1, ..., A_k, B_k are undetermined coefficients which are determined by the methods given above. For $k=1$, the fraction (5) is integrated directly; for $k>1$, use is made of the *reduction method*; here, it is first advisable to represent the quadratic trinomial x^2+px+q in the form $\left(x+\frac{p}{2}\right)^2+\left(q-\frac{p^2}{4}\right)$ and make the substitution $x+\frac{p}{2}=z$.

Example 3. Find

$$\int \frac{x+1}{(x^2+4x+5)^2}\,dx=I.$$

Solution. Since

$$x^2+4x+5=(x+2)^2+1,$$

then, putting $x+2=z$, we get

$$I=\int \frac{z-1}{(z^2+1)^2}\,dz=\int \frac{z\,dz}{(z^2+1)^2}-\int \frac{(1+z^2)-z^2}{(z^2+1)^2}\,dz=$$

$$=-\frac{1}{2(z^2+1)}-\int \frac{dz}{z^2+1}+\int z\,d\left[-\frac{1}{2(z^2+1)}\right]=-\frac{1}{2(z^2+1)}-$$

$$-\arctan z-\frac{z}{2(z^2+1)}+\frac{1}{2}\arctan z=-\frac{z+1}{2(z^2+1)}-$$

$$-\frac{1}{2}\arctan z+C=-\frac{x+3}{2(x^2+4x+5)}-\frac{1}{2}\arctan(x+2)+C.$$

2°. The Ostrogradsky method. If $Q(x)$ has multiple roots, then

$$\int \frac{P(x)}{Q(x)}\,dx=\frac{X(x)}{Q_1(x)}+\int \frac{Y(x)}{Q_2(x)}\,dx, \qquad (6)$$

where $Q_1(x)$ is the greatest common divisor of the polynomial $Q(x)$ and its derivative $Q'(x)$;

$$Q_2(x)=Q(x):Q_1(x);$$

$X(x)$ and $Y(x)$ are polynomials with undetermined coefficients, whose degrees are, respectively, less by unity than those of $Q_1(x)$ and $Q_2(x)$.

The undetermined coefficients of the polynomials $X(x)$ and $Y(x)$ are computed by differentiating the identity (6).

Example 4. Find

$$\int \frac{dx}{(x^3-1)^2}\,.$$

Solution.

$$\int \frac{dx}{(x^3-1)^2} = \frac{Ax^2+Bx+C}{x^3-1} + \int \frac{Dx^2+Ex+F}{x^3-1}\,dx$$

Differentiating this identity, we get

$$\frac{1}{(x^3-1)^2} = \frac{(2Ax+B)(x^3-1)-3x^2(Ax^2+Bx+C)}{(x^3-1)^2} + \frac{Dx^2+Ex+F}{x^3-1}$$

or

$$1=(2Ax+B)(x^3-1)-3x^2(Ax^2+Bx+C)+(Dx^2+Ex+F)(x^3-1).$$

Equating the coefficients of the respective degrees of x, we will have:

$$D=0;\quad E-A=0;\quad F-2B=0;\quad D+3C=0;\quad E+2A=0;\quad B+F=-1;$$

whence

$$A=0;\quad B=-\frac{1}{3};\quad C=0;\quad D=0;\quad E=0;\quad F=-\frac{2}{3}$$

and, consequently,

$$\int \frac{dx}{(x^3-1)^2} = -\frac{1}{3}\frac{x}{x^3-1} - \frac{2}{3}\int \frac{dx}{x^3-1} \tag{7}$$

To compute the integral on the right of (7), we decompose the fraction $\frac{1}{x^3-1}$ into partial fractions:

$$\frac{1}{x^3-1} = \frac{L}{x-1} + \frac{Mx+N}{x^2+x+1},$$

that is,

$$1=L(x^2+x+1)+Mx(x-1)+N(x-1). \tag{8}$$

Putting $x=1$, we get $L=\frac{1}{3}$.

Equating the coefficients of identical degrees of x on the right and left of (8), we find

$$L+M=0;\quad L-N=1,$$

or

$$M=-\frac{1}{3};\quad N=-\frac{2}{3}.$$

Therefore,

$$\int \frac{dx}{x^3-1} = \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{3}\int \frac{x+2}{x^2+x+1}\,dx =$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln(x^2+x+1) - \frac{1}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}} + C$$

and

$$\int \frac{dx}{(x^3-1)^2} = -\frac{x}{3(x^3-1)} + \frac{1}{9}\ln\frac{x^2+x+1}{(x-1)^2} + \frac{2}{3\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}} + C.$$

Find the following integrals:

1280. $\int \frac{dx}{(x+a)(x+b)}$.

1281. $\int \frac{x^2-5x+9}{x^2-5x+6}\,dx$.

1282. $\int \frac{dx}{(x+1)(x+2)(x+3)}$.

1283. $\int \frac{2x^2+41x-91}{(x-1)(x+3)(x-4)}\,dx$.

1284. $\int \frac{5x^3+2}{x^3-5x^2+4x}\,dx.$

1285. $\int \frac{dx}{x(x+1)^2}.$

1286. $\int \frac{x^3-1}{4x^3-x}\,dx.$

1287. $\int \frac{x^4-6x^3+12x^2+6}{x^3-6x^2+12x-8}\,dx.$

1288. $\int \frac{5x^2+6x+9}{(x-3)^2(x+1)^2}\,dx.$

1289. $\int \frac{x^2-8x+7}{(x^2-3x-10)^2}\,dx.$

1290. $\int \frac{2x-3}{(x^2-3x+2)^3}\,dx.$

1291. $\int \frac{x^3+x+1}{x(x^2+1)}\,dx.$

1292. $\int \frac{x^4}{x^4-1}\,dx.$

1293. $\int \frac{dx}{(x^2-4x+3)(x^2+4x+5)}.$

1294. $\int \frac{dx}{x^3+1}.$

1295. $\int \frac{dx}{x^4+1}.$

1296. $\int \frac{dx}{x^4+x^2+1}.$

1297. $\int \frac{dx}{(1+x^2)^2}.$

1298. $\int \frac{3x+5}{(x^2+2x+2)^2}\,dx.$

1299. $\int \frac{dx}{(x+1)(x^2+x+1)^2}.$

1300. $\int \frac{x^3+1}{(x^2-4x+5)^2}\,dx.$

Applying Ostrogradsky's method, find the following integrals:

1301. $\int \frac{dx}{(x+1)^2(x^2+1)^2}.$

1302. $\int \frac{dx}{(x^4-1)^2}.$

1303. $\int \frac{dx}{(x^2+1)^4}.$

1304. $\int \frac{x^4-2x^2+2}{(x^2-2x+2)^2}\,dx.$

Applying different procedures, find the integrals:

1305. $\int \frac{x^5}{(x^3+1)(x^3+8)}\,dx.$

1306. $\int \frac{x^7+x^3}{x^{12}-2x^4+1}\,dx.$

1307. $\int \frac{x^2-x+14}{(x-4)^3(x-2)}\,dx.$

1308. $\int \frac{dx}{x^4(x^3+1)^2}$

1309. $\int \frac{dx}{x^3-4x^2+5x-2}.$

1310*. $\int \frac{dx}{x(x^7+1)}.$

1311. $\int \frac{dx}{x(x^5+1)^2}.$

1312. $\int \frac{dx}{(x^2+2x+2)(x^2+2x+5)}.$

1313. $\int \frac{x^2\,dx}{(x-1)^{10}}.$

1314. $\int \frac{dx}{x^8+x^6}.$

Sec. 6. Integrating Certain Irrational Functions

1°. Integrals of the form

$$\int R\left[x, \left(\frac{ax+b}{cx+d}\right)^{\frac{p_1}{q_1}}, \left(\frac{ax+b}{cx+d}\right)^{\frac{p_2}{q_2}}, \ldots\right]dx, \qquad (1)$$

where R is a rational function and p_1, q_1, p_2, q_2 are whole numbers.

Integrals of form (1) are found by the substitution

$$\frac{ax+b}{cx+d}=z^n,$$

where n is the least common multiple of the numbers $q_1, q_2, \ldots$

Example 1. Find

$$\int \frac{dx}{\sqrt{2x-1}-\sqrt[4]{2x-1}}.$$

Solution. The substitution $2x-1=z^4$ leads to an integral of the form

$$\int \frac{dx}{\sqrt{2x-1}-\sqrt[4]{2x-1}}=\int \frac{2z^3dz}{z^2-z}=2\int \frac{z^2dz}{z-1}=$$

$$=2\int \left(z+1+\frac{1}{z-1}\right)dz=(z+1)^2+2\ln|z-1|+C=$$

$$=\left(1+\sqrt[4]{2x-1}\right)^2+\ln\left(\sqrt[4]{2x-1}-1\right)^2+C.$$

Find the integrals:

1315. $\int \frac{x^3}{\sqrt{x-1}}dx.$

1316. $\int \frac{x\,dx}{\sqrt[3]{ax+b}}.$

1317. $\int \frac{dx}{\sqrt{x+1}+\sqrt{(x+1)^3}}.$

1318. $\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}.$

1319. $\int \frac{\sqrt[6]{x}-1}{\sqrt[3]{x}+1}dx.$

1320. $\int \frac{\sqrt{x+1}+2}{(x+1)^2-\sqrt{x+1}}dx.$

1321. $\int \frac{\sqrt{x}}{x+2}dx.$

1322. $\int \frac{dx}{(2-x)\sqrt{1-x}}.$

1323. $\int x\sqrt{\frac{x-1}{x+1}}dx.$

1324. $\int \sqrt[3]{\frac{x+1}{x-1}}dx.$

1325. $\int \frac{x+3}{x^2\sqrt{2x+3}}dx.$

2°. Integrals of the form

$$\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}dx, \qquad (2)$$

where $P_n(x)$ is a polynomial of degree n

Put

$$\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}dx=Q_{n-1}(x)\sqrt{ax^2+bx+c}+\lambda\int \frac{dx}{\sqrt{ax^2+bx+c}}, \qquad (3)$$

where $Q_{n-1}(x)$ is a polynomial of degree $(n-1)$ with undetermined coefficients and λ is a number.

The coefficients of the polynomial $Q_{n-1}(x)$ and the number λ are found by differentiating identity (3).

Example 2.

$$\int x^2\sqrt{x^2+4}\,dx=\int\frac{x^4+4x^2}{\sqrt{x^2+4}}\,dx=$$

$$=(Ax^3+Bx^2+Cx+D)\sqrt{x^2+4}+\lambda\int\frac{dx}{\sqrt{x^2+4}}.$$

Whence

$$\frac{x^4+4x^2}{\sqrt{x^2+4}}=(3Ax^2+2Bx+C)\sqrt{x^2+4}+\frac{(Ax^3+Bx^2+Cx+D)\,x}{\sqrt{x^2+4}}+\frac{\lambda}{\sqrt{x^2+4}}.$$

Multiplying by $\sqrt{x^2+4}$ and equating the coefficients of identical degrees of x, we obtain

$$A=\frac{1}{4};\quad B=0;\quad C=\frac{1}{2};\quad D=0;\quad \lambda=-2$$

Hence,

$$\int x^2\sqrt{x^2+4}\,dx=\frac{x^3+2x}{4}\sqrt{x^2+4}-2\ln\left(x+\sqrt{x^2+4}\right)+C.$$

3°. Integrals of the form

$$\int\frac{dx}{(x-\alpha)^n\sqrt{ax^2+bx+c}}. \tag{4}$$

They are reduced to integrals of the form (2) by the substitution:

$$\frac{1}{x-\alpha}=t.$$

Find the integrals:

1326. $\int\frac{x^2dx}{\sqrt{x^2-x+1}}$.

1327. $\int\frac{x^5}{\sqrt{1-x^2}}\,dx$.

1328. $\int\frac{x^6}{\sqrt{1+x^2}}\,dx$.

1329. $\int\frac{dx}{x^5\sqrt{x^2-1}}$.

1330. $\int\frac{dx}{(x+1)^3\sqrt{x^2+2x}}$.

1331. $\int\frac{x^2+x+1}{x\sqrt{x^2-x+1}}\,dx$.

4°. Integrals of the binomial differentials

$$\int x^m(a+bx^n)^p\,dx, \tag{5}$$

where m, n and p are rational numbers.

Chebyshev's conditions. The integral (5) can be expressed in terms of a finite combination of elementary functions only in the following three cases:

1) if p is a whole number;

2) if $\frac{m+1}{n}$ is a whole number. Here, we make the substitution $a+bx^n=z^s$, where s is the denominator of the fraction p;

3) if $\frac{m+1}{n}+p$ is a whole number. Here, use is made of the substitution $ax^{-n}+b=z^s$.

Example 3. Find

$$\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}\,dx=I.$$

Solution. Here, $m=-\frac{1}{2}$; $n=\frac{1}{4}$; $p=\frac{1}{3}$; $\frac{m+1}{n}=\frac{-\frac{1}{2}+1}{\frac{1}{4}}=2$. Hence, we have here Case 2 integrability.

The substitution

$$1+x^{\frac{1}{4}}=z^3$$

yields $x=(z^3-1)^4$; $dx=12z^2(z^3-1)^3\,dz$ Therefore,

$$I=\int x^{-\frac{1}{2}}\left(1+x^{\frac{1}{4}}\right)^{\frac{1}{3}}dx=12\int\frac{z^3(z^3-1)^3}{(z^3-1)^2}\,dz=$$

$$=12\int(z^6-z^3)\,dz=\frac{12}{7}z^7-3z^4+C,$$

where $z=\sqrt[3]{1+\sqrt[4]{x}}$.

Find the integrals:

1332. $\int x^3(1+2x^2)^{-\frac{3}{2}}\,dx$.

1333. $\int \frac{dx}{\sqrt[4]{1+x^4}}$.

1334. $\int \frac{dx}{x^4\sqrt{1+x^2}}$.

1335. $\int \frac{dx}{x\sqrt[3]{1+x^5}}$.

1336. $\int \frac{dx}{x^2(2+x^3)^{\frac{5}{3}}}$.

1337. $\int \frac{dx}{\sqrt{x^3}\,\sqrt[3]{1+\sqrt[4]{x^3}}}$.

Sec. 7. Integrating Trigonometric Functions

1°. Integrals of the form

$$\int \sin^m x\cos^n x\,dx=I_{m,n}, \tag{1}$$

where m and n are integers.

1) If $m=2k+1$ is an odd positive number, then we put

$$I_{m,n}=-\int\sin^{2k}x\cos^n x\,d(\cos x)=-\int(1-\cos^2 x)^k\cos^n x\,d(\cos x).$$

We do the same if n is an odd positive number.

Example 1.

$$\int\sin^{10}x\cos^3x\,dx=\int\sin^{10}x(1-\sin^2 x)\,d(\sin x)=$$

$$=\frac{\sin^{11}x}{11}-\frac{\sin^{13}x}{13}+C.$$

2) If m and n are even positive numbers, then the integrand (1) is transformed by means of the formulas

$$\sin^2 x = \frac{1}{2}(1-\cos 2x), \quad \cos^2 x = \frac{1}{2}(1+\cos 2x),$$

$$\sin x \cos x = \frac{1}{2}\sin 2x.$$

Example 2. $\int \cos^2 3x \sin^4 3x\, dx = \int (\cos 3x \sin 3x)^2 \sin^2 3x\, dx =$

$$= \int \frac{\sin^2 6x}{4}\,\frac{1-\cos 6x}{2}\, dx = \frac{1}{8}\int (\sin^2 6x - \sin^2 6x \cos 6x)\, dx =$$

$$= \frac{1}{8}\int \left(\frac{1-\cos 12x}{2} = \sin^2 6x \cos 6x\right) dx =$$

$$= \frac{1}{8}\left(\frac{x}{2} - \frac{\sin 12x}{24} - \frac{1}{18}\sin^3 6x\right) + C.$$

3) If $m=-\mu$ and $n=-\nu$ are integral negative numbers of identical parity, then

$$I_{m,n} = \int \frac{dx}{\sin^{\mu} x \cos^{\nu} x} = \int \operatorname{cosec}^{\mu} x \sec^{\nu-2} x\, d(\tan x) =$$

$$= \int \left(1+\frac{1}{\tan^2 x}\right)^{\frac{\mu}{2}} (1+\tan^2 x)^{\frac{\nu-2}{2}}\, d(\tan x) = \int \frac{(1+\tan^2 x)^{\frac{\mu+\nu}{2}-1}}{\tan^{\mu} x}\, d(\tan x).$$

In particular, the following integrals reduce to this case:

$$\int \frac{dx}{\sin^{\mu} x} = \frac{1}{2^{\mu-1}} \int \frac{d\left(\frac{x}{2}\right)}{\sin^{\mu}\frac{x}{2}\cos^{\mu}\frac{x}{2}} \quad \text{and} \quad \int \frac{dx}{\cos^{\nu} x} = \int \frac{d\left(x+\frac{\pi}{2}\right)}{\sin^{\nu}\left(x+\frac{\pi}{2}\right)}.$$

Example 3. $\int \frac{dx}{\cos^4 x} = \int \sec^2 x\, d(\tan x) = \int (1+\tan^2 x)\, d(\tan x) =$

$$= \tan x + \frac{1}{3}\tan^3 x + C.$$

Example 4. $\int \frac{dx}{\sin^3 x} = \frac{1}{2^3}\int \frac{dx}{\sin^3\frac{x}{2}\cos^3\frac{x}{2}} = \frac{1}{8}\int \tan^{-3}\frac{x}{2}\sec^6\frac{x}{2}\, dx =$

$$= \frac{1}{8}\int \frac{\left(1+\tan^2\frac{x}{2}\right)^2}{\tan^3\frac{x}{2}}\sec^2\frac{x}{2}\, dx = \frac{2}{8}\int \left[\tan^{-3}\frac{x}{2} + \frac{2}{\tan\frac{x}{2}} + \right.$$

$$\left. + \tan\frac{x}{2}\right] d\left(\tan\frac{x}{2}\right) = \frac{1}{4}\left[-\frac{1}{2\tan^2\frac{x}{2}} + 2\ln\left|\tan\frac{x}{2}\right| + \frac{\tan^2\frac{x}{2}}{2}\right] + C.$$

4) Integrals of the form $\int \tan^m x\,dx$ (or $\int \cot^m x\,dx$), where m is an integral positive number, are evaluated by the formula

$$\tan^2 x = \sec^2 x - 1$$

(or, respectively, $\cot^2 x = \operatorname{cosec}^2 x - 1$).

Example 5. $\int \tan^4 x\,dx = \int \tan^2 x(\sec^2 x - 1)\,dx = \frac{\tan^3 x}{3} - \int \tan^2 x\,dx =$
$= \frac{\tan^3 x}{3} - \int (\sec^2 x - 1)\,dx = \frac{\tan^3 x}{3} - \tan x + x + C.$

5) In the general case, integrals $I_{m,n}$ of the form (1) are evaluated by means of *reduction formulas* that are usually derived by integration by parts.

Example 6. $\int \frac{dx}{\cos^3 x} = \int \frac{\sin^2 x + \cos^2 x}{\cos^3 x}\,dx =$
$= \int \sin x \cdot \frac{\sin x}{\cos^3 x}\,dx + \int \frac{dx}{\cos x} = \sin x \cdot \frac{1}{2\cos^2 x} - \frac{1}{2}\int \frac{\cos x}{\cos^2 x}\,dx + \int \frac{dx}{\cos x} =$
$= \frac{\sin x}{2\cos^2 x} + \frac{1}{2}\ln|\tan x + \sec x| + C.$

Find the integrals:

1338. $\int \cos^2 x\,dx.$

1339. $\int \sin^3 x\,dx.$

1340. $\int \sin^2 x \cos^3 x\,dx.$

1341. $\int \sin^3 \frac{x}{2} \cos^5 \frac{x}{2}\,dx.$

1342. $\int \frac{\cos^5 x}{\sin^3 x}\,dx.$

1343. $\int \sin^4 x\,dx.$

1344. $\int \sin^2 x \cos^2 x\,dx.$

1345. $\int \sin^2 x \cos^4 x\,dx.$

1346. $\int \cos^6 3x\,dx.$

1347. $\int \frac{dx}{\sin^4 x}.$

1348. $\int \frac{dx}{\cos^6 x}.$

1349. $\int \frac{\cos^2 x}{\sin^6 x}\,dx.$

1350. $\int \frac{dx}{\sin^2 x \cos^4 x}.$

1351. $\int \frac{dx}{\sin^5 x \cos^3 x}.$

1352. $\int \frac{dx}{\sin \frac{x}{2} \cos^3 \frac{x}{2}}.$

1353. $\int \frac{\sin\left(x + \frac{\pi}{4}\right)}{\sin x \cos x}\,dx.$

1354. $\int \frac{dx}{\sin^5 x}.$

1355. $\int \sec^5 4x\,dx.$

1356. $\int \tan^2 5x\,dx.$

1357. $\int \cot^3 x\,dx.$

1358. $\int \cot^4 x\,dx.$

1359. $\int \left(\tan^3 \frac{x}{3} + \tan^4 \frac{x}{4}\right) dx.$

1360. $\int x \sin^2 x^2\,dx.$

1361. $\int \frac{\cos^2 x}{\sin^4 x}\,dx.$

1362. $\int \sin^3 x \sqrt[3]{\cos x}\,dx.$

1363. $\int \frac{dx}{\sqrt{\sin x \cos^3 x}}.$

1364. $\int \frac{dx}{\sqrt{\tan x}}.$

2°. **Integrals of the form** $\int \sin mx \cos nx\,dx$, $\int \sin mx \sin nx\,dx$ **and** $\int \cos mx \cos nx\,dx$. In these cases the following formulas are used:

1) $\sin mx \cos nx = \frac{1}{2}[\sin(m+n)x + \sin(m-n)x]$;

2) $\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]$;

3) $\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]$.

Example 7. $\int \sin 9x \sin x\,dx = \int \frac{1}{2}[\cos 8x - \cos 10x]\,dx =$
$= \frac{1}{16}\sin 8x - \frac{1}{20}\sin 10x + C$.

Find the integrals:

1365. $\int \sin 3x \cos 5x\,dx$.

1366. $\int \sin 10x \sin 15x\,dx$.

1367. $\int \cos\frac{x}{2}\cos\frac{x}{3}\,dx$.

1368. $\int \sin\frac{x}{3}\sin\frac{2x}{3}\,dx$.

1369. $\int \cos(ax+b)\cos(ax-b)\,dx$.

1370. $\int \sin\omega t \sin(\omega t + \varphi)\,dt$.

1371. $\int \cos x \cos^2 3x\,dx$.

1372. $\int \sin x \sin 2x \sin 3x\,dx$.

3°. **Integrals of the form**

$$\int R(\sin x, \cos x)\,dx, \tag{2}$$

where R is a rational function.

1) By means of substitution

$$\tan\frac{x}{2} = t,$$

whence

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad dx = \frac{2dt}{1+t^2},$$

integrals of form (2) are reduced to integrals of rational functions by the new variable t.

Example 8. Find

$$\int \frac{dx}{1+\sin x+\cos x} = I.$$

Solution. Putting $\tan\frac{x}{2} = t$, we will have

$$I = \int \frac{\frac{2dt}{1+t^2}}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}} = \int \frac{dt}{1+t} = \ln|1+t| + C = \ln\left|1+\tan\frac{x}{2}\right| + C.$$

2) If we have the identity

$$R(-\sin x,\ -\cos x) \equiv R(\sin x,\ \cos x),$$

then we can use the substitution $\tan x = t$ to reduce the integral (2) to a rational form.

Here,

$$\sin x = \frac{t}{\sqrt{1+t^2}},\quad \cos x = \frac{1}{\sqrt{1+t^2}}$$

and

$$x = \arctan t,\ dx = \frac{dt}{1+t^2}.$$

Example 9. Find

$$\int \frac{dx}{1+\sin^2 x} = I. \tag{3}$$

Solution. Putting

$$\tan x = t,\quad \sin^2 x = \frac{t^2}{1+t^2},\quad dx = \frac{dt}{1+t^2},$$

we will have

$$I = \int \frac{dt}{(1+t^2)\left(1+\frac{t^2}{1+t^2}\right)} = \int \frac{dt}{1+2t^2} = \frac{1}{\sqrt{2}} \int \frac{d(t\sqrt{2})}{1+(t\sqrt{2})^2} =$$

$$= \frac{1}{\sqrt{2}} \arctan(t\sqrt{2}) + C = \frac{1}{\sqrt{2}} \arctan(\sqrt{2}\tan x) + C.$$

We note that the integral (3) is evaluated faster if the numerator and denominator of the fraction are first divided by $\cos^2 x$.

In individual cases, it is useful to apply artificial procedures (see, for example, 1379).

Find the integrals:

1373. $\int \frac{dx}{3+5\cos x}$.

1374. $\int \frac{dx}{\sin x+\cos x}$.

1375. $\int \frac{\cos x}{1+\cos x}\, dx$.

1376. $\int \frac{\sin x}{1-\sin x}\, dx$.

1377. $\int \frac{dx}{8-4\sin x+7\cos x}$.

1378. $\int \frac{dx}{\cos x+2\sin x+3}$.

1379**. $\int \frac{3\sin x+2\cos x}{2\sin x+3\cos x}\, dx$.

1380. $\int \frac{1+\tan x}{1-\tan x}\, dx$.

1381*. $\int \frac{dx}{1+3\cos^2 x}$.

1382*. $\int \frac{dx}{3\sin^2 x+5\cos^2 x}$.

1383*. $\int \frac{dx}{\sin^2 x+3\sin x\cos x-\cos^2 x}$.

1384*. $\int \frac{dx}{\sin^2 x-5\sin x\cos x}$.

1385. $\int \frac{\sin x}{(1-\cos x)^3}\, dx$.

1386. $\int \frac{\sin 2x}{1+\sin^2 x}\, dx$.

1387. $\int \frac{\cos 2x}{\cos^4 x+\sin^4 x}\, dx$.

1388. $\int \frac{\cos x}{\sin^2 x-6\sin x+5}\, dx$.

1389*. $\int \frac{dx}{(2-\sin x)(3-\sin x)}$.

1390*. $\int \frac{1-\sin x+\cos x}{1+\sin x-\cos x}\, dx$.

Sec. 8. Integration of Hyperbolic Functions

Integration of hyperbolic functions is completely analogous to the integration of trigonometric functions.

The following basic formulas should be remembered:

1) $\cosh^2 x - \sinh^2 x = 1$;

2) $\sinh^2 x = \frac{1}{2}(\cosh 2x - 1)$;

3) $\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$;

4) $\sinh x \cosh x = \frac{1}{2}\sinh 2x$.

Example 1. Find

$$\int \cosh^2 x\, dx.$$

Solution. We have

$$\int \cosh^2 x\, dx = \int \frac{1}{2}(\cosh 2x + 1)\, dx = \frac{1}{4}\sinh 2x + \frac{1}{2}x + C.$$

Example 2. Find

$$\int \cosh^3 x\, dx.$$

Solution. We have

$$\int \cosh^3 x\, dx = \int \cosh^2 x d(\sinh x) = \int (1 + \sinh^2 x)\, d(\sinh x) = \sinh x + \frac{\sinh^3 x}{3} + C.$$

Find the integrals:

1391. $\int \sinh^3 x\, dx$.

1392. $\int \cosh^4 x\, dx$.

1393. $\int \sinh^3 x \cosh x\, dx$.

1394. $\int \sinh^2 x \cosh^2 x\, dx$.

1395. $\int \frac{dx}{\sinh x \cosh^2 x}$.

1396. $\int \frac{dx}{\sinh^2 x \cosh^2 x}$.

1397. $\int \tanh^3 x\, dx$.

1398. $\int \coth^4 x\, dx$.

1399. $\int \frac{dx}{\sinh^2 x + \cosh^2 x}$.

1400. $\int \frac{dx}{2\sinh x + 3\cosh x}$.

1401*. $\int \frac{dx}{\tanh x - 1}$.

1402. $\int \frac{\sinh x\, dx}{\sqrt{\cosh 2x}}$.

Sec. 9. Using Trigonometric and Hyperbolic Substitutions for Finding Integrals of the Form

$$\int R(x, \sqrt{ax^2 + bx + c})\, dx, \qquad (1)$$

where R is a rational function.

Transforming the quadratic trinomial ax^2+bx+c into a sum or difference of squares, the integral (1) becomes reducible to one of the following types of integrals:

1) $\int R(z, \sqrt{m^2-z^2})\,dz$;

2) $\int R(z, \sqrt{m^2+z^2})\,dz$;

3) $\int R(z, \sqrt{z^2-m^2})\,dz$.

The latter integrals are, respectively, taken by means of substitutions:

1) $z=m\sin t$ or $z=m\tanh t$,
2) $z=m\tan t$ or $z=m\sinh t$,
3) $z=m\sec t$ or $z=m\cosh t$.

Example 1. Find

$$\int \frac{dx}{(x+1)^2\sqrt{x^2+2x+2}}=I.$$

Solution. We have

$$x^2+2x+2=(x+1)^2+1.$$

Putting $x+1=\tan z$, we then have $dx=\sec^2 z\,dz$ and

$$I=\int \frac{dx}{(x+1)^2\sqrt{(x+1)^2+1}}=\int \frac{\sec^2 z\,dz}{\tan^2 z\sec z}=\int \frac{\cos z}{\sin^2 z}\,dz=$$

$$=-\frac{1}{\sin z}+C=\frac{\sqrt{x^2+2x+2}}{x+1}+C.$$

Example 2. Find

$$\int x\sqrt{x^2+x+1}\,dx=I.$$

Solution. We have

$$x^2+x+1=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}.$$

Putting

$$x+\frac{1}{2}=\frac{\sqrt{3}}{2}\sinh t \quad \text{and} \quad dx=\frac{\sqrt{3}}{2}\cosh t\,dt,$$

we get

$$I=\int \left(\frac{\sqrt{3}}{2}\sinh t-\frac{1}{2}\right)\frac{\sqrt{3}}{2}\cosh t\cdot\frac{\sqrt{3}}{2}\cosh t\,dt=$$

$$=\frac{3\sqrt{3}}{8}\int \sinh t\cosh^2 t\,dt-\frac{3}{8}\int \cosh^2 t\,dt=$$

$$=\frac{3\sqrt{3}}{8}\,\frac{\cosh^3 t}{3}-\frac{3}{8}\left(\frac{1}{2}\sinh t\cosh t+\frac{1}{2}t\right)+C.$$

Since

$$\sinh t=\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right),\ \cosh t=\frac{2}{\sqrt{3}}\sqrt{x^2+x+1}$$

and

$$t=\ln\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)+\ln\frac{2}{\sqrt{3}},$$

we finally have

$$I=\frac{1}{3}(x^2+x+1)^{\frac{3}{2}}-\frac{1}{4}\left(x+\frac{1}{2}\right)\sqrt{x^2+x+1}-$$
$$-\frac{3}{16}\ln\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)+C.$$

Find the integrals:

1403. $\int\sqrt{3-2x-x^2}\,dx.$

1404. $\int\sqrt{2+x^2}\,dx.$

1405. $\int\frac{x^2}{\sqrt{9+x^2}}\,dx.$

1406. $\int\sqrt{x^2-2x+2}\,dx.$

1407. $\int\sqrt{x^2-4}\,dx.$

1408. $\int\sqrt{x^2+x}\,dx.$

1409. $\int\sqrt{x^2-6x-7}\,dx.$

1410. $\int(x^2+x+1)^{\frac{3}{2}}\,dx.$

1411. $\int\frac{dx}{(x-1)\sqrt{x^2-3x+2}}.$

1412. $\int\frac{dx}{(x^2-2x+5)^{\frac{3}{2}}}.$

1413. $\int\frac{dx}{(1+x^2)\sqrt{1-x^2}}.$

1414. $\int\frac{dx}{(1-x^2)\sqrt{1+x^2}}.$

Sec. 10. Integration of Various Transcendental Functions

Find the integrals:

1415. $\int(x^2+1)^2e^{2x}\,dx.$

1416. $\int x^2\cos^2 3x\,dx.$

1417. $\int x\sin x\cos 2x\,dx.$

1418. $\int e^{2x}\sin^2 x\,dx.$

1419. $\int e^x\sin x\sin 3x\,dx.$

1420. $\int xe^x\cos x\,dx.$

1421. $\int\frac{dx}{e^{2x}+e^x-2}.$

1422. $\int\frac{dx}{\sqrt{e^{2x}+e^x+1}}.$

1423. $\int x^2\ln\frac{1+x}{1-x}\,dx.$

1424. $\int\ln^2(x+\sqrt{1+x^2})\,dx.$

1425. $\int x\arccos(5x-2)\,dx.$

1426. $\int\sin x\sinh x\,dx.$

Sec. 11. Using Reduction Formulas

Derive the reduction formulas for the following integrals:

1427. $I_n=\int\frac{dx}{(x^2+a^2)^n}$; find I_2 and I_3.

1428. $I_n=\int\sin^n x\,dx$; find I_4 and I_5.

1429. $I_n=\int \frac{dx}{\cos^n x}$; find I_3 and I_4.

1430. $I_n=\int x^n e^{-x}\,dx$; find I_{10}.

Sec. 12. Miscellaneous Examples on Integration

1431. $\int \frac{dx}{2x^2-4x+9}$.

1432. $\int \frac{x-5}{x^2-2x+2}\,dx$.

1433. $\int \frac{x^3}{x^2+x+\frac{1}{2}}\,dx$.

1434. $\int \frac{dx}{x(x^2+5)}$.

1435. $\int \frac{dx}{(x+2)^2(x+3)^2}$.

1436. $\int \frac{dx}{(x+1)^2(x^2+1)}$.

1437. $\int \frac{dx}{(x^2+2)^2}$.

1438. $\int \frac{dx}{x^4-2x^2+1}$.

1439. $\int \frac{x\,dx}{(x^2-x+1)^3}$.

1440. $\int \frac{3-4x}{(1-2\sqrt{x})^2}\,dx$.

1441. $\int \frac{(\sqrt{x}+1)^2}{x^3}\,dx$.

1442. $\int \frac{dx}{\sqrt{x^2+x+1}}$.

1443. $\int \frac{1-\sqrt[3]{2x}}{\sqrt{2x}}\,dx$.

1444. $\int \frac{dx}{(\sqrt[3]{x^2}+\sqrt[3]{x})^2}$.

1445. $\int \frac{2x+1}{\sqrt{(4x^2-2x+1)^3}}\,dx$.

1446. $\int \frac{dx}{\sqrt[4]{5-x}+\sqrt{5-x}}$.

1447. $\int \frac{x^2}{\sqrt{(x^2-1)^3}}\,dx$.

1448. $\int \frac{x\,dx}{(1+x^2)\sqrt{1-x^4}}$.

1449. $\int \frac{x\,dx}{\sqrt{1-2x^2-x^4}}$.

1450. $\int \frac{x+1}{(x^2+1)^{\frac{3}{2}}}\,dx$.

1451*. $\int \frac{dx}{(x^2+4x)\sqrt{4-x^2}}$.

1452. $\int \sqrt{x^2-9}\,dx$.

1453. $\int \sqrt{x-4x^2}\,dx$.

1454. $\int \frac{dx}{x\sqrt{x^2+x+1}}$.

1455. $\int x\sqrt{x^2+2x+2}\,dx$.

1456. $\int \frac{dx}{x^4\sqrt{x^2-1}}$.

1457. $\int \frac{dx}{x\sqrt{1-x^3}}$.

1458. $\int \frac{dx}{\sqrt[3]{1+x^3}}$.

1459. $\int \frac{5x}{\sqrt{1+x^4}}\,dx$.

1460. $\int \cos^4 x\,dx$.

1461. $\int \frac{dx}{\cos x \sin^5 x}$.

1462. $\int \frac{1+\sqrt{\cot x}}{\sin^2 x}\,dx$.

1463. $\int \frac{\sin^3 x}{\sqrt[5]{\cos^3 x}}\,dx$.

1464. $\int \operatorname{cosec}^5 5x\,dx$.

1465. $\int \frac{\sin^2 x}{\cos^6 x}\,dx$.

1466. $\int \sin\left(\frac{\pi}{4}-x\right)\sin\left(\frac{\pi}{4}+x\right) dx.$

1467. $\int \tan^3\left(\frac{x}{2}+\frac{\pi}{4}\right) dx.$

1468. $\int \frac{dx}{2\sin x+3\cos x-5}.$

1469. $\int \frac{dx}{2+3\cos^2 x}.$

1470. $\int \frac{dx}{\cos^2 x+2\sin x\cos x+2\sin^2 x}.$

1471. $\int \frac{dx}{\sin x\sin 2x}.$

1472. $\int \frac{dx}{(2+\cos x)(3+\cos x)}.$

1473. $\int \frac{\sec^2 x}{\sqrt{\tan^2 x+4\tan x+1}} dx.$

1474. $\int \frac{\cos ax}{\sqrt{a^2+\sin^2 ax}} dx.$

1475. $\int \frac{x\,dx}{\cos^2 3x}.$

1476. $\int x\sin^2 x\,dx.$

1477. $\int x^2 e^{x^3} dx.$

1478. $\int xe^{2x} dx.$

1479. $\int x^2 \ln\sqrt{1-x}\,dx.$

1480. $\int \frac{x\arctan x}{\sqrt{1+x^2}} dx.$

1481. $\int \sin^2\frac{x}{2}\cos\frac{3x}{2} dx.$

1482. $\int \frac{dx}{(\sin x+\cos x)^2}.$

1483. $\int \frac{dx}{(\tan x+1)\sin^2 x}.$

1484. $\int \sinh x\cosh x\,dx.$

1485. $\int \frac{\sinh\sqrt{1-x}}{\sqrt{1-x}} dx.$

1486. $\int \frac{\sinh x\cosh x}{\sinh^2 x+\cosh^2 x} dx.$

1487. $\int \frac{x}{\sinh^2 x} dx.$

1488. $\int \frac{dx}{e^{2x}-2e^x}.$

1489. $\int \frac{e^x}{e^{2x}-6e^x+13} dx.$

1490. $\int \frac{e^{2x}}{(e^x+1)^{\frac{1}{4}}} dx.$

1491. $\int \frac{2^x}{1-4^x} dx.$

1492. $\int (x^2-1)\,10^{-2x} dx.$

1493. $\int \sqrt{e^x+1}\,dx.$

1494. $\int \frac{\arctan x}{x^2} dx.$

1495. $\int x^3 \arcsin\frac{1}{x} dx.$

1496. $\int \cos(\ln x)\,dx.$

1497. $\int (x^2-3x)\sin 5x\,dx.$

1498. $\int x\arctan(2x+3)\,dx.$

1499. $\int \arcsin\sqrt{x}\,dx.$

1500. $\int |x|\,dx.$

Chapter V

DEFINITE INTEGRALS

Sec. 1. The Definite Integral as the Limit of a Sum

1°. Integral sum. Let a function $f(x)$ be defined on an interval $a \leqslant x \leqslant b$, and $a = x_0 < x_1 < \ldots < x_n = b$ is an arbitrary partition of this interval into n subintervals (Fig. 37). A sum of the form

$$S_n = \sum_{i=0}^{n-1} f(\xi_i)\,\Delta x_i, \tag{1}$$

where

$$x_i \leqslant \xi_i \leqslant x_{i+1};\quad \Delta x_i = x_{i+1} - x_i;$$
$$i = 0,\ 1,\ 2,\ \ldots\ (n-1),$$

is called the *integral sum* of the function $f(x)$ on $[a, b]$. Geometrically, S_n is the algebraic area of a step-like figure (see Fig. 37).

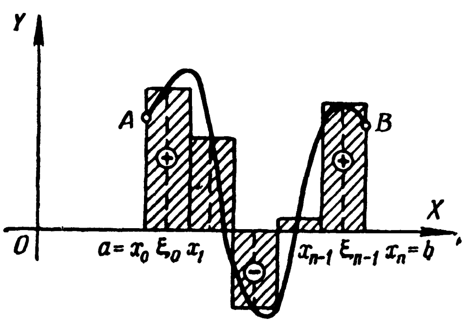

Fig. 37

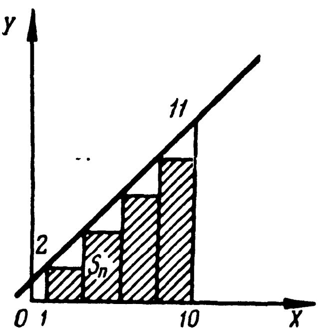

Fig. 38

2°. The definite integral. The limit of the sum S_n, provided that the number of subdivisions n tends to infinity, and the largest of them, Δx_i, to zero, is called the *definite integral* of the function $f(x)$ within the limits from $x = a$ to $x = b$; that is,

$$\lim_{\max \Delta x_i \to 0} \sum_{i=0}^{n-1} f(\xi_i)\,\Delta x_i = \int_a^b f(x)\,dx. \tag{2}$$

If the function $f(x)$ is continuous on $[a, b]$, it is integrable on $[a, b]$; i.e., the limit of (2) exists and is independent of the mode of partition of the interval of integration $[a, b]$ into subintervals and is independent of the choice of points ξ_i in these subintervals. Geometrically, the definite integral (2) is the algebraic sum of the areas of the figures that make up the curvilinear trapezoid $aABb$, in which the areas of the parts located above the x-axis are plus, those below the x-axis, minus (Fig. 37).

The definitions of integral sum and definite integral are naturally generalized to the case of an interval $[a, b]$, where $a > b$.

Example 1. Form the integral sum S_n for the function

$$f(x) = 1 + x$$

on the interval [1,10] by dividing the interval into n equal parts and choosing points ξ_i that coincide with the left end-points of the subintervals $[x_i, x_{i+1}]$. What is the $\lim\limits_{n \to \infty} S_n$ equal to?

Solution. Here, $\Delta x_i = \frac{10-1}{n} = \frac{9}{n}$ and $\xi_i = x_i = x_0 + i\Delta x_i = 1 + \frac{9i}{n}$. Whence $(\xi_i) = 1 + 1 + \frac{9i}{n} = 2 + \frac{9i}{n}$. Hence (Fig. 38),

$$S_n = \sum_{i=0}^{n-1} f(\xi_i)\,\Delta x_i = \sum_{i=0}^{n-1}\left(2 + \frac{9i}{n}\right)\frac{9}{n} = \frac{18}{n}n + \frac{81}{n^2}(0 + 1 + \ldots + n - 1) =$$

$$= 18 + \frac{81}{n^2}\,\frac{n(n-1)}{2} = 18 + \frac{81}{2}\left(1 - \frac{1}{n}\right) = 58\,\frac{1}{2} - \frac{81}{2n},$$

$$\lim_{n \to \infty} S_n = 58\,\frac{1}{2}.$$

Example 2. Find the area bounded by an arc of the parabola $y = x^2$, the x-axis, and the ordinates $x = 0$, and $x = a$ $(a > 0)$.

Solution. Partition the base a into n equal parts $\Delta x = \frac{a}{n}$. Choosing the value of the function at the beginning of each subinterval, we will have

$$y_1 = 0;\ y_2 = \left(\frac{a}{n}\right)^2;\ y_3 = \left[2\left(\frac{a}{n}\right)^2\right];\ \ldots;$$

$$y_n = \left[(n-1)\frac{a}{n}\right]^2.$$

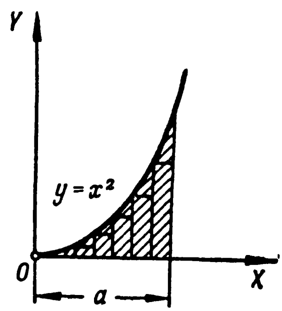

Fig. 39

The areas of the rectangles are obtained by multiplying each y_k by the base $\Delta x = \frac{a}{n}$ (Fig. 39). Summing, we get the area of the step-like figure

$$S_n = \frac{a}{n}\left(\frac{a}{n}\right)^2 [1 + 2^2 + 3^2 + \ldots + (n-1)^2].$$

Using the formula for the sum of the squares of integers,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6},$$

we find

$$S_n = \frac{a^3 n (n-1)(2n-1)}{6n^3},$$

and, passing to the limit, we obtain

$$S = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{a^3 (n-1) n (2n-1)}{6n^3} = \frac{a^3}{3}.$$

Evaluate the following definite integrals, regarding them as the limits of appropriate integral sums:

1501. $\int_a^b dx.$

1502. $\int_0^T (v_0 + gt)\,dt,$

v_0 and g are constant.

1503. $\int_{-2}^{1} x^2 dx.$

1504. $\int_0^{10} 2^x dx.$

1505*. $\int_1^5 x^3 dx.$

1506*. Find the area of a curvilinear trapezoid bounded by the hyperbola

$$y = \frac{1}{x},$$

by two ordinates: $x = a$ and $x = b$ $(0 < a < b)$, and the x-axis.

1507*. Find

$$f(x) = \int_0^x \sin t\,dt.$$

Sec. 2. Evaluating Definite Integrals by Means of Indefinite Integrals

1°. A definite integral with variable upper limit. If a function $f(t)$ is continuous on an interval $[a, b]$, then the function

$$F(x) = \int_a^x f(t)\,dt$$

is the antiderivative of the function $f(x)$; that is,

$$F'(x) = f(x) \quad \text{for} \quad a \leqslant x \leqslant b.$$

2°. The Newton-Leibniz formula. If $F'(x) = f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

The antiderivative $F(x)$ is computed by finding the indefinite integral

$$\int f(x)\,dx = F(x) + C.$$

Example 1. Find the integral

$$\int_{-1}^{3} x^4\,dx.$$

Solution. $\int\limits_{-1}^{3} x^4\,dx = \frac{x^5}{5}\Big|_{-1}^{3} = \frac{3^5}{5} - \frac{(-1)^5}{5} = 48\frac{4}{5}.$

1508. Let

$$I = \int_a^b \frac{dx}{\ln x} \quad (b > a > 1).$$

Find

$$1)\ \frac{dI}{da};\quad 2)\ \frac{dI}{db}.$$

Find the derivatives of the following functions:

1509. $F(x) = \int\limits_1^x \ln t\,dt \quad (x > 0)$.

1510. $F(x) = \int\limits_x^0 \sqrt{1+t^4}\,dt$.

1511. $F(x) = \int\limits_x^{x^2} e^{-t^2}\,dt$.

1512. $I = \int\limits_{\frac{1}{x}}^{\sqrt{x}} \cos(t^2)\,dt \quad (x > 0)$.

1513. Find the points of the extremum of the function

$$y = \int_0^x \frac{\sin t}{t}\,dt \text{ in the region } x > 0.$$

Applying the Newton-Leibniz formula, find the integrals:

1514. $\int\limits_0^1 \frac{dx}{1+x}$.

1515. $\int\limits_{-2}^{-1} \frac{dx}{x^3}$.

1516. $\int\limits_{-x}^{x} e^t\,dt$.

1517. $\int\limits_0^x \cos t\,dt$.

Using definite integrals, find the limits of the sums:

1518.** $\lim\limits_{n\to\infty}\left(\frac{1}{n^2} + \frac{2}{n^2} + \ldots + \frac{n-1}{n^2}\right)$.

1519.** $\lim\limits_{n\to\infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n}\right)$.

1520. $\lim\limits_{n\to\infty} \frac{1^p + 2^p + \ldots + n^p}{n^{p+1}} \quad (p > 0)$.

Evaluate the integrals:

1521. $\int_1^2 (x^2-2x+3)\,dx$.

1522. $\int_0^8 (\sqrt{2x}+\sqrt[3]{x})\,dx$.

1523. $\int_1^4 \frac{1+\sqrt{y}}{y^2}\,dy$.

1524. $\int_2^6 \sqrt{x-2}\,dx$.

1525. $\int_0^{-3} \frac{dx}{\sqrt{25+3x}}$.

1526. $\int_{-2}^{-3} \frac{dx}{x^2-1}$.

1527. $\int_0^1 \frac{x\,dx}{x^2+3x+2}$.

1528. $\int_{-1}^1 \frac{y^5\,dy}{y+2}$.

1529. $\int_0^1 \frac{dx}{x^2+4x+5}$.

1530. $\int_3^4 \frac{dx}{x^2-3x+2}$.

1531. $\int_0^1 \frac{z^3}{z^8+1}\,dz$.

1532. $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sec^2\alpha\,d\alpha$.

1533. $\int_0^{\frac{\sqrt{2}}{2}} \frac{dx}{\sqrt{1-x^2}}$.

1534. $\int_2^5 \frac{dx}{\sqrt{5+4x-x^2}}$.

1535. $\int_0^1 \frac{y^2\,dy}{\sqrt{y^6+4}}$.

1536. $\int_0^{\frac{\pi}{4}} \cos^2\alpha\,d\alpha$.

1537. $\int_0^{\frac{\pi}{2}} \sin^3\varphi\,d\varphi$.

1538. $\int_e^{e^2} \frac{dx}{x\ln x}$.

1539. $\int_1^e \frac{\sin(\ln x)}{x}\,dx$.

1540. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan x\,dx$.

1541. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cot^4\varphi\,d\varphi$.

1542. $\int_0^1 \frac{e^x}{1+e^{2x}}\,dx$.

1543. $\int_0^1 \cosh x\,dx$.

1544. $\int_{\ln 2}^{\ln 3} \frac{dx}{\cosh^2 x}$.

1545. $\int_0^{\pi} \sinh^2 x\,dx$.

Sec. 3. Improper Integrals

1°. Integrals of unbounded functions. If a function $f(x)$ is not bounded in any neighbourhood of a point c of an interval $[a, b]$ and is continuous for $a \leqslant x < c$ and $c < x \leqslant b$, then by definition we put

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0} \int_a^{c-\varepsilon} f(x)\,dx + \lim_{\varepsilon \to 0} \int_{c+\varepsilon}^b f(x)\,dx. \tag{1}$$

If the limits on the right side of (1) exist and are finite, the improper integral is called *convergent*, otherwise it is *divergent*. When $c=a$ or $c=b$, the definition is correspondingly simplified.

If there is a continuous function $F(x)$ on $[a, b]$ such that $F'(x)=f(x)$ when $x \neq c$ *(generalized antiderivative)*, then

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{2}$$

If $|f(x)| \leqslant F(x)$ when $a \leqslant x \leqslant b$ and $\int_a^b F(x)\,dx$ converges, then the integral (1) also converges *(comparison test)*.

If $f(x) \geqslant 0$ and $\lim\limits_{x \to c} f(x)\,|c-x|^m = A \neq \infty$, $A \neq 0$, i.e., $f(x) \sim \dfrac{A}{|c-x|^m}$ when $x \to c$, then 1) for $m < 1$ the integral (1) converges, 2) for $m \geqslant 1$ the integral (1) diverges.

2°. Integrals with infinite limits. If the function $f(x)$ is continuous when $a \leqslant x < \infty$, then we assume

$$\int_a^{\infty} f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx \tag{3}$$

and depending on whether there is a finite limit or not on the right of (3), the respective integral is called *convergent* or *divergent*.

Similarly,

$$\int_{-\infty}^b f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)\,dx = \lim_{\substack{a \to -\infty \\ b \to +\infty}} \int_a^b f(x)\,dx.$$

If $|f(x)| \leqslant F(x)$ and the integral $\int_a^{\infty} F(x)\,dx$ converges, then the integral (3) converges as well.

If $f(x) \geqslant 0$ and $\lim\limits_{x \to \infty} f(x)\,x^m = A \neq \infty$, $A \neq 0$, i.e., $f(x) \sim \dfrac{A}{x^m}$ when $x \to \infty$, then 1) for $m > 1$ the integral (3) converges, 2) for $m \leqslant 1$ the integral (3) diverges.

Example 1.

$$\int_{-1}^{1} \frac{dx}{x^2} = \lim_{\varepsilon \to 0} \int_{-1}^{-\varepsilon} \frac{dx}{x^2} + \lim_{\varepsilon \to 0} \int_{\varepsilon}^{1} \frac{dx}{x^2} = \lim_{\varepsilon \to 0} \left(\frac{1}{\varepsilon} - 1\right) + \lim_{\varepsilon \to 0} \left(\frac{1}{\varepsilon} - 1\right) = \infty$$

and the integral diverges.

Example 2.

$$\int_0^{\infty} \frac{dx}{1+x^2} = \lim_{b \to \infty} \int_0^b \frac{dx}{1+x^2} = \lim_{b \to \infty} (\arctan b - \arctan 0) = \frac{\pi}{2}.$$

Example 3. Test the convergence of the *probability integral*

$$\int_0^{\infty} e^{-x^2}\, dx. \tag{4}$$

Solution. We put

$$\int_0^{\infty} e^{-x^2}\, dx = \int_0^1 e^{-x^2}\, dx + \int_1^{\infty} e^{-x^2}\, dx.$$

The first of the two integrals on the right is not an improper integral, while the second one converges, since $e^{-x^2} \leqslant e^{-x}$ when $x \geqslant 1$ and

$$\int_1^{\infty} e^{-x}\, dx = \lim_{b \to \infty} \int_1^b e^{-x}\, dx = \lim_{b \to \infty} (-e^{-b} + e^{-1}) = e^{-1};$$

hence, the integral (4) converges.

Example 4. Test the following integral for convergence:

$$\int_1^{\infty} \frac{dx}{\sqrt{x^3+1}}. \tag{5}$$

Solution. When $x \to +\infty$, we have

$$\frac{1}{\sqrt{x^3+1}} = \frac{1}{\sqrt{x^3\left(1+\frac{1}{x^3}\right)}} = \frac{1}{x^{\frac{3}{2}}} \frac{1}{\sqrt{1+\frac{1}{x^3}}} \sim \frac{1}{x^{\frac{3}{2}}}.$$

Since the integral

$$\int_1^{\infty} \frac{dx}{x^{\frac{3}{2}}}$$

converges, our integral (5) likewise converges.

Example 5. Test for convergence the elliptic integral

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}}. \tag{6}$$

Solution. The point of discontinuity of the integrand is $x=1$. Applying the Lagrange formula we get

$$\frac{1}{\sqrt{1-x^4}}=\frac{1}{\sqrt{(1-x)\cdot 4x_1^3}}=\frac{1}{(1-x)^{\frac{1}{4}}}\cdot\frac{1}{2x_1^{\frac{3}{2}}},$$

where $x<x_1<1$. Hence, for $x\to 1$ we have

$$\frac{1}{\sqrt{1-x^4}}\sim\frac{1}{2}\left(\frac{1}{1-x}\right)^{\frac{1}{4}}.$$

Since the integral

$$\int\limits_0^1\left(\frac{1}{1-x}\right)^{\frac{1}{4}}dx$$

converges, the given integral (6) converges as well.

Evaluate the improper integrals (or establish their divergence):

1546. $\int\limits_0^1\frac{dx}{\sqrt{x}}$.

1547. $\int\limits_{-1}^2\frac{dx}{x}$.

1548. $\int\limits_0^1\frac{dx}{x^p}$.

1549. $\int\limits_0^3\frac{dx}{(x-1)^2}$.

1550. $\int\limits_0^1\frac{dx}{\sqrt{1-x^2}}$.

1551. $\int\limits_1^\infty\frac{dx}{x}$.

1552. $\int\limits_1^\infty\frac{dx}{x^2}$.

1553. $\int\limits_1^\infty\frac{dx}{x^p}$.

1554. $\int\limits_{-\infty}^\infty\frac{dx}{1+x^2}$.

1555. $\int\limits_{-\infty}^\infty\frac{dx}{x^2+4x+9}$.

1556. $\int\limits_0^\infty\sin x\,dx$.

1557. $\int\limits_0^{\frac{1}{2}}\frac{dx}{x\ln x}$.

1558. $\int\limits_0^{\frac{1}{2}}\frac{dx}{x\ln^2 x}$.

1559. $\int\limits_a^\infty\frac{dx}{x\ln x}$ $(a>1)$.

1560. $\int\limits_a^\infty\frac{dx}{x\ln^2 x}$ $(a>1)$.

1561. $\int\limits_0^{\frac{\pi}{2}}\cot x\,dx$.

1562. $\int\limits_0^{\infty} e^{-kx}\,dx \quad (k>0)$.

1563. $\int\limits_0^{\infty} \frac{\arctan x}{x^2+1}\,dx$.

1564. $\int\limits_2^{\infty} \frac{dx}{(x^2-1)^2}$.

1565. $\int\limits_0^{\infty} \frac{dx}{x^3+1}$.

1566. $\int\limits_0^{1} \frac{dx}{x^3-5x^2}$.

Test the convergence of the following integrals:

1567. $\int\limits_0^{100} \frac{dx}{\sqrt[3]{x}+2\sqrt[4]{x}+x^3}$.

1568. $\int\limits_1^{+\infty} \frac{dx}{2x+\sqrt[3]{x^2+1}+5}$.

1569. $\int\limits_{-1}^{\infty} \frac{dx}{x^2+\sqrt[3]{x^4+1}}$.

1570. $\int\limits_0^{\infty} \frac{x\,dx}{\sqrt{x^5+1}}$.

1571. $\int\limits_0^{1} \frac{dx}{\sqrt[3]{1-x^4}}$.

1572. $\int\limits_1^{2} \frac{dx}{\ln x}$.

1573. $\int\limits_{\frac{\pi}{2}}^{\infty} \frac{\sin x}{x^2}\,dx$.

1574*. Prove that the Euler integral of the first kind *(beta-function)*

$$B(p,\ q)=\int\limits_0^1 x^{p-1}(1-x)^{q-1}\,dx$$

converges when $p>0$ and $q>0$.

1575*. Prove that the Euler integral of the second kind *(gamma-function)*

$$\Gamma(p)=\int\limits_0^{\infty} x^{p-1}e^{-x}\,dx$$

converges for $p>0$.

Sec. 4. Change of Variable in a Definite Integral

If a function $f(x)$ is continuous over $a \leqslant x \leqslant b$ and $x=\varphi(t)$ is a function continuous together with its derivative $\varphi'(t)$ over $\alpha \leqslant t \leqslant \beta$, where $a=\varphi(\alpha)$ and $b=\varphi(\beta)$, and $f[\varphi(t)]$ is defined and continuous on the interval $\alpha \leqslant t \leqslant \beta$,

then

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f[\varphi(t)]\,\varphi'(t)\,dt.$$

Example 1. Find

$$\int_0^a x^2 \sqrt{a^2-x^2\,dx} \qquad (a>0).$$

Solution. We put

$$x = a\sin t;$$
$$dx = a\cos t\,dt.$$

Then $t = \arc\sin\frac{x}{a}$ and, consequently, we can take $\alpha = \arcsin 0 = 0$, $\beta = \arcsin 1 = \frac{\pi}{2}$. Therefore, we shall have

$$\int_0^a x^2\sqrt{a^2-x^2}\,dx = \int_0^{\frac{\pi}{2}} a^2\sin^2 t\,\sqrt{a^2-a^2\sin^2 t}\;a\cos t\,dt =$$
$$= a^4\int_0^{\frac{\pi}{2}} \sin^2 t\cos^2 t\,dt = \frac{a^4}{4}\int_0^{\frac{\pi}{2}}\sin^2 2t\,dt = \frac{a^4}{8}\int_0^{\frac{\pi}{2}}(1-\cos 4t)\,dt =$$
$$= \frac{a^4}{8}\left(t-\frac{1}{4}\sin 4t\right)\Bigg|_0^{\frac{\pi}{2}} = \frac{\pi a^4}{16}.$$

1576. Can the substitution $x = \cos t$ be made in the integral

$$\int_0^2 \sqrt[3]{1-x^2}\,dx?$$

Transform the following definite integrals by means of the indicated substitutions:

1577. $\int_1^3 \sqrt{x+1}\,dx$, $x = 2t-1$.

1578. $\int_{\frac{1}{2}}^{1} \frac{dx}{\sqrt{1-x^4}}$, $x = \sin t$.

1579. $\int_{\frac{3}{4}}^{\frac{4}{3}} \frac{dx}{\sqrt{x^2+1}}$, $x = \sinh t$.

1580. $\int_0^{\frac{\pi}{2}} f(x)\,dx$, $x = \arctan t$.

1581. For the integral

$$\int_a^b f(x)\,dx \quad (b>a)$$

indicate an integral linear substitution

$$x=\alpha t+\beta,$$

as a result of which the limits of integration would be 0 and 1, respectively.

Applying the indicated substitutions, evaluate the following integrals:

1582. $\int\limits_0^4 \frac{dx}{1+\sqrt{x}}$, $\quad x=t^2$.

1583. $\int\limits_3^{29} \frac{(x-2)^{2/3}}{(x-2)^{2/3}+3}\,dx$, $\quad x-2=z^3$.

1584. $\int\limits_0^{\ln 2} \sqrt{e^x-1}\,dx$, $\quad e^x-1=z^2$.

1585. $\int\limits_0^{\pi} \frac{dt}{3+2\cos t}$, $\quad \tan\frac{t}{2}=z$.

1586. $\int\limits_0^{\frac{\pi}{2}} \frac{dx}{1+a^2\sin^2 x}$, $\quad \tan x=t$.

Evaluate the following integrals by means of appropriate substitutions:

1587. $\int\limits_{\frac{\sqrt{2}}{2}}^{1} \frac{\sqrt{1-x^2}}{x^2}\,dx$.

1588. $\int\limits_1^2 \frac{\sqrt{x^2-1}}{x}\,dx$.

1589. $\int\limits_0^{\ln 5} \frac{e^x\sqrt{e^x-1}}{e^x+3}\,dx$.

1590. $\int\limits_0^5 \frac{dx}{2x+\sqrt{3x+1}}$.

Evaluate the integrals:

1591. $\int\limits_1^3 \frac{dx}{x\sqrt{x^2+5x+1}}$.

1592. $\int\limits_{-1}^{1} \frac{dx}{(1+x^2)^2}$.

1593. $\int\limits_0^a \sqrt{ax-x^2}\,dx$.

1594. $\int\limits_0^{2\pi} \frac{dx}{5-3\cos x}$.

1595. Prove that if $f(x)$ is an even function, then

$$\int\limits_{-a}^{a} f(x)\,dx=2\int\limits_0^a f(x)\,dx.$$

But if $f(x)$ is an odd function, then

$$\int_{-a}^{a} f(x)\,dx = 0.$$

1596. Show that

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = 2\int_{0}^{\infty} e^{-x^2}dx = \int_{0}^{\infty} \frac{e^{-x}}{\sqrt{x}}\,dx.$$

1597. Show that

$$\int_{0}^{1} \frac{dx}{\arccos x} = \int_{0}^{\frac{\pi}{2}} \frac{\sin x}{x}\,dx.$$

1598. Show that

$$\int_{0}^{\frac{\pi}{2}} f(\sin x)\,dx = \int_{0}^{\frac{\pi}{2}} f(\cos x)\,dx.$$

Sec. 5. Integration by Parts

If the functions $u(x)$ and $v(x)$ are continuously differentiable on the interval $[a, b]$, then

$$\int_{a}^{b} u(x)\,v'(x)\,dx = u(x)\,v(x)\Big|_{a}^{b} - \int_{a}^{b} v(x)\,u'(x)\,dx. \qquad (1)$$

Applying the formula for integration by parts, evaluate the following integrals:

1599. $\int_{0}^{\frac{\pi}{2}} x\cos x\,dx.$

1600. $\int_{1}^{e} \ln x\,dx.$

1601. $\int_{0}^{1} x^3e^{2x}dx.$

1602. $\int_{0}^{\pi} e^{x}\sin x\,dx.$

1603. $\int_{0}^{\infty} xe^{-x}dx.$

1604. $\int_{0}^{\infty} e^{-ax}\cos bx\,dx \quad (a>0).$

1605. $\int_{0}^{\infty} e^{-ax}\sin bx\,dx \quad (a>0).$

1606.** Show that for the gamma-function (see Example 1575) the following reduction formula holds true:

$$\Gamma(p+1)=p\Gamma(p) \quad (p>0).$$

From this derive that $\Gamma(n+1)=n!$, if n is a natural number.

1607. Show that for the integral

$$I_n=\int_0^{\frac{\pi}{2}} \sin^n x\,dx=\int_0^{\frac{\pi}{2}} \cos^n x\,dx$$

the *reduction formula*

$$I_n=\frac{n-1}{n}I_{n-2}$$

holds true.

Find I_n, if n is a natural number. Using the formula obtained, evaluate I_9 and I_{10}.

1608. Applying repeated integration by parts, evaluate the integral (see Example 1574)

$$\mathrm{B}(p, q)=\int_0^1 x^{p-1}(1-x)^{q-1}\,dx,$$

where p and q are positive integers.

1609*. Express the following integral in terms of B (beta-function):

$$I_{n,m}=\int_0^{\frac{\pi}{2}} \sin^m x\cos^n x\,dx,$$

if m and n are nonnegative integers.

Sec. 6. Mean-Value Theorem

1°. Evaluation of integrals. If $f(x)\leqslant F(x)$ for $a\leqslant x\leqslant b$, then

$$\int_a^b f(x)\,dx\leqslant\int_a^b F(x)\,dx. \qquad (1)$$

If $f(x)$ and $\varphi(x)$ are continuous for $a\leqslant x\leqslant b$ and, besides, $\varphi(x)\geqslant 0$, then

$$m\int_a^b \varphi(x)\,dx\leqslant\int_a^b f(x)\varphi(x)\,dx\leqslant M\int_a^b \varphi(x)\,dx, \qquad (2)$$

where m is the smallest and M is the largest value of the function $f(x)$ on the interval $[a, b]$.

In particular, if $\varphi(x) \equiv 1$, then

$$m(b-a) \leqslant \int_a^b f(x)\,dx \leqslant M(b-a). \tag{3}$$

The inequalities (2) and (3) may be replaced, respectively, by their equivalent equalities:

$$\int_a^b f(x)\,\varphi(x)\,dx = f(c)\int_a^b \varphi(x)\,dx$$

and

$$\int_a^b f(x)\,dx = f(\xi)\,(b-a),$$

where c and ξ are certain numbers lying between a and b.

Example 1. Evaluate the integral

$$I = \int_0^{\frac{\pi}{2}} \sqrt{1+\frac{1}{2}\sin^2 x}\,dx.$$

Solution. Since $0 \leqslant \sin^2 x \leqslant 1$, we have

$$\frac{\pi}{2} < I < \frac{\pi}{2}\sqrt{\frac{3}{2}},$$

that is,

$$1.57 < I < 1.91.$$

2°. The mean value of a function. The number

$$\mu = \frac{1}{b-a}\int_a^b f(x)\,dx$$

is called the *mean value* of the function $f(x)$ on the interval $a \leqslant x \leqslant b$.

1610*. Determine the signs of the integrals without evaluating them:

a) $\int_{-1}^{2} x^3\,dx$;

b) $\int_0^{\pi} x\cos x\,dx$;

c) $\int_0^{2\pi} \frac{\sin x}{x}\,dx$.

1611. Determine (without evaluating) which of the following integrals is greater:

a) $\int\limits_0^1 \sqrt{1+x^2}\,dx$ or $\int\limits_0^1 dx$;

b) $\int\limits_0^1 x^2 \sin^2 x\,dx$ or $\int\limits_0^1 x \sin^2 x\,dx$;

c) $\int\limits_1^2 e^{x^2}dx$ or $\int\limits_1^2 e^x dx$.

Find the mean values of the functions on the indicated intervals:

1612. $f(x)=x^2$, $0 \leqslant x \leqslant 1$.

1613. $f(x)=a+b\cos x$, $-\pi \leqslant x \leqslant \pi$.

1614. $f(x)=\sin^2 x$, $0 \leqslant x \leqslant \pi$.

1615. $f(x)=\sin^4 x$, $0 \leqslant x \leqslant \pi$.

1616. Prove that $\int\limits_0^1 \frac{dx}{\sqrt{2+x-x^2}}$ lies between $\frac{2}{3} \approx 0.67$ and $\frac{1}{\sqrt{2}} \approx$ ≈ 0.70. Find the exact value of this integral.

Evaluate the integrals:

1617. $\int\limits_0^1 \sqrt{4+x^2}\,dx$.

1618. $\int\limits_{-1}^{+1} \frac{dx}{8+x^3}$.

1619. $\int\limits_0^{2\pi} \frac{dx}{10+3\cos x}$.

1620*. $\int\limits_0^{\frac{\pi}{4}} x\sqrt{\tan x}$.

1621. $\int\limits_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin x}{x}\,dx$.

1622. Integrating by parts, prove that

$$0 < \int\limits_{100\pi}^{200\pi} \frac{\cos x}{x}\,dx < \frac{1}{100\pi}.$$

Sec. 7. The Areas of Plane Figures

1°. Area in rectangular coordinates. If a continuous curve is defined in rectangular coordinates by the equation $y=f(x)$ $[f(x)\geqslant 0]$, the area of the curvilinear trapezoid bounded by this curve, by two vertical lines at the

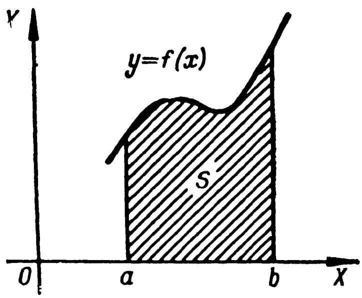

Fig. 40

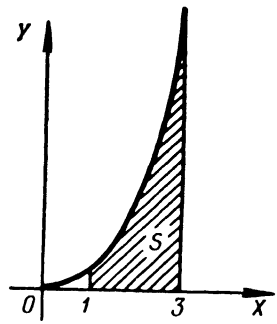

Fig. 41

points $x=a$ and $x=b$ and by a segment of the x-axis $a\leqslant x\leqslant b$ (Fig. 40), is given by the formula

$$S=\int_a^b f(x)\,dx. \tag{1}$$

Example 1. Compute the area bounded by the parabola $y=\frac{x^2}{2}$, the straight lines $x=1$ and $x=3$, and the x-axis (Fig. 41).

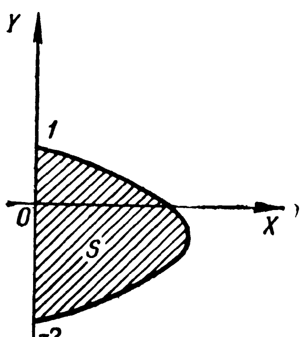

Fig. 42

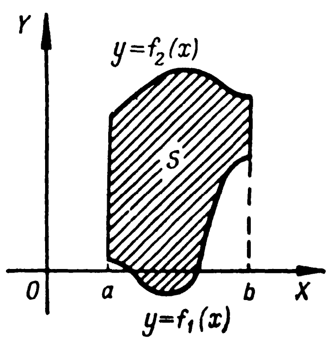

Fig. 43

Solution. The sought-for area is expressed by the integral

$$S=\int_1^3 \frac{x^2}{2}\,dx=4\frac{1}{3}.$$

Example 2. Evaluate the area bounded by the curve $x=2-y-y^2$ and the y-axis (Fig. 42).

Solution. Here, the roles of the coordinate axes are changed and so the sought-for area is expressed by the integral

$$S=\int_{-2}^{1}(2-y-y^2)\,dy=4\frac{1}{2},$$

where the limits of integration $y_1=-2$ and $y_2=1$ are found as the ordinates of the points of intersection of the curve with the y-axis.

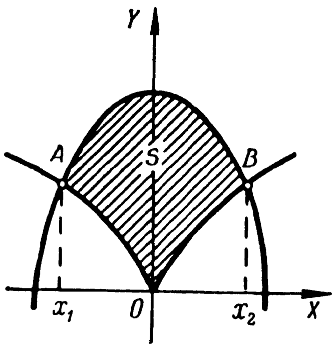

Fig. 44

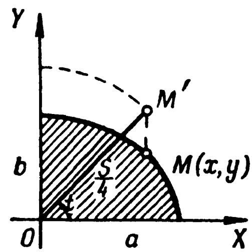

Fig. 45

In the more general case, if the area S is bounded by two continuous curves $y=f_1(x)$ and $y=f_2(x)$ and by two vertical lines $x=a$ and $x=b$, where $f_1(x)\leqslant f_2(x)$ when $a\leqslant x\leqslant b$ (Fig. 43), we will then have:

$$S=\int_{a}^{b}[f_2(x)-f_1(x)]\,dx. \tag{2}$$

Example 3. Evaluate the area S contained between the curves

$$y=2-x^2 \text{ and } y^3=x^2 \tag{3}$$

(Fig. 44).

Solution. Solving the set of equations (3) simultaneously, we find the limits of integration: $x_1=-1$ and $x_2=1$. By virtue of formula (2), we obtain

$$S=\int_{-1}^{1}(2-x^2-x^{2/3})\,dx=\left(2x-\frac{x^3}{3}-\frac{3}{5}x^{\frac{5}{3}}\right)_{-1}^{1}=2\frac{2}{15}.$$

If the curve is defined by equations in parametric form $x=\varphi(t)$, $y=\psi(t)$, then the area of the curvilinear trapezoid bounded by this curve, by two

vertical lines ($x=a$ and $x=b$), and by a segment of the x-axis is expressed by the integral

$$S=\int_{t_1}^{t_2} \psi(t)\,\varphi'(t)\,dt,$$

where t_1 and t_2 are determined from the equations
$a=\varphi(t_1)$ and $b=\varphi(t_2)$ [$\psi(t)\geqslant 0$ on the interval $[t_1, t_2]$].

Example 4. Find the area of the ellipse (Fig. 45) by using its parametric equations

$$\begin{cases} x=a\cos t, \\ y=b\sin t. \end{cases}$$

Solution. Due to the symmetry, it is sufficient to compute the area of a quadrant and then multiply the result by four. If in the equation $x=a\cos t$ we first put $x=0$ and then $x=a$, we get the limits of integration $t_1=\frac{\pi}{2}$ and $t_2=0$. Therefore,

$$\frac{1}{4}S=\int_{\frac{\pi}{2}}^{0} b\sin a\,(-\sin t)\,dt=ab\int_{0}^{\frac{\pi}{2}}\sin^2 t\,dt=\frac{\pi ab}{4}$$

and, hence, $S=\pi ab$.

2°. The area in polar coordinates. If a curve is defined in polar coordinates by the equation $r=f(\varphi)$, then the area of the sector AOB (Fig. 46), bounded by an arc of the curve, and by two radius vectors OA and OB,

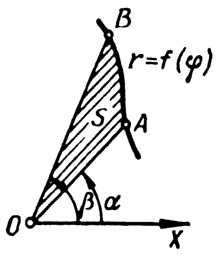

Fig. 46

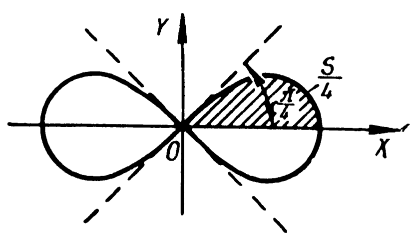

Fig. 47

which correspond to the values $\varphi_1=\alpha$ and $\varphi_2=\beta$, is expressed by the integral

$$S=\frac{1}{2}\int_{\alpha}^{\beta}[f(\varphi)]^2\,d\varphi.$$

Example 5. Find the area contained inside Bernoulli's lemniscate $r^2=a^2\cos 2\varphi$ (Fig. 47).

Solution. By virtue of the symmetry of the curve we determine first one quadrant of the sought-for area:

$$\frac{1}{4}S=\frac{1}{2}\int_{0}^{\frac{\pi}{4}} a^2\cos 2\varphi\, d\varphi=\frac{a^2}{2}\left[\frac{1}{2}\sin 2\varphi\right]_0^{\frac{\pi}{4}}=\frac{a^2}{4}.$$

Whence $S=a^2$.

1623. Compute the area bounded by the parabola $y=4x-x^2$ and the x-axis.

1624. Compute the area bounded by the curve $y=\ln x$, the x-axis and the straight line $x=e$.

1625*. Find the area bounded by the curve $y=x(x-1)(x-2)$ and the x-axis.

1626. Find the area bounded by the curve $y^3=x$, the straight line $y=1$ and the vertical line $x=8$.

1627. Compute the area bounded by a single half-wave of the sinusoidal curve $y=\sin x$ and the x-axis.

1628. Compute the area contained between the curve $y=\tan x$, the x-axis and the straight line $x=\frac{\pi}{3}$.

1629. Find the area contained between the hyperbola $xy=m^2$, the vertical lines $x=a$ and $x=3a$ $(a>0)$ and the x-axis.

1630. Find the area contained between the witch of Agnesi $y=\frac{a^3}{x^2+a^2}$ and the x-axis.

1631. Compute the area of the figure bounded by the curve $y=x^3$, the straight line $y=8$ and the y-axis.

1632. Find the area bounded by the parabolas $y^2=2px$ and $x^2=2py$.

1633. Evaluate the area bounded by the parabola $y=2x-x^2$ and the straight line $y=-x$.

1634. Compute the area of a segment cut off by the straight line $y=3-2x$ from the parabola $y=x^2$.

1635. Compute the area contained between the parabolas $y=x^2$, $y=\frac{x^2}{2}$ and the straight line $y=2x$.

1636. Compute the area contained between the parabolas $y=\frac{x^2}{3}$ and $y=4-\frac{2}{3}x^2$.

1637. Compute the area contained between the witch of Agnesi $y=\frac{1}{1+x^2}$ and the parabola $y=\frac{x^2}{2}$.

1638. Compute the area bounded by the curves $y=e^x$, $y=e^{-x}$ and the straight line $x=1$.

1639. Find the area of the figure bounded by the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and the straight line $x=2a$.

1640*. Find the entire area bounded by the astroid

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}.$$

1641. Find the area between the catenary

$$y=a\cosh\frac{x}{a},$$

the y-axis and the straight line $y=\frac{a}{2e}(e^2+1)$.

1642. Find the area bounded by the curve $a^2y^2=x^2(a^2-x^2)$.

1643. Compute the area contained within the curve

$$\left(\frac{x}{5}\right)^2+\left(\frac{y}{4}\right)^{\frac{2}{3}}=1.$$

1644. Find the area between the equilateral hyperbola $x^2-y^2=$ $=9$, the x-axis and the diameter passing through the point (5,4).

1645. Find the area between the curve $y=\frac{1}{x^2}$, the x-axis, and the ordinate $x=1$ $(x>1)$.

1646*. Find the area bounded by the cissoid $y^2=\frac{x^3}{2a-x}$ and its asymptote $x=2a$ $(a>0)$.

1647*. Find the area between the strophoid $y^2=\frac{x(x-a)^2}{2a-x}$ and its asymptote $(a>0)$.

1648. Compute the area of the two parts into which the circle $x^2+y^2=8$ is divided by the parabola $y^2=2x$.

1649. Compute the area contained between the circle $x^2+y^2=16$ and the parabola $x^2=12(y-1)$.

1650. Find the area contained within the astroid

$$x=a\cos^3 t;\quad y=b\sin^3 t.$$

1651. Find the area bounded by the x-axis and one arc of the cycloid

$$\begin{cases} x=a(t-\sin t), \\ y=a(1-\cos t). \end{cases}$$

1652. Find the area bounded by one branch of the trochoid

$$\begin{cases} x=at-b\sin t, \\ y=a-b\cos t \end{cases} \quad (0<b\leqslant a)$$

and a tangent to it at its lower points.

1653. Find the area bounded by the cardioid

$$\begin{cases} x = a(2\cos t - \cos 2t), \\ y = a(2\sin t - \sin 2t). \end{cases}$$

1654*. Find the area of the loop of the folium of Descartes

$$x = \frac{3at}{1+t^3}; \quad y = \frac{3at^2}{1+t^3}.$$

1655*. Find the entire area of the cardioid $r = a(1+\cos\varphi)$.

1656*. Find the area contained between the first and second turns of Archimedes' spiral, $r = a\varphi$ (Fig. 48).

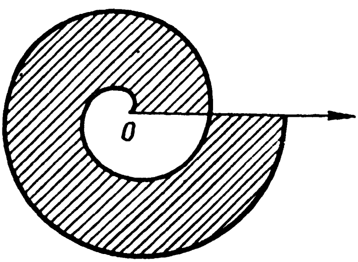

Fig. 48

1657. Find the area of one of the leaves of the curve $r = a\cos 2\varphi$.

1658. Find the entire area bounded by the curve $r^2 = a^2 \sin 4\varphi$.

1659*. Find the area bounded by the curve $r = a\sin 3\varphi$.

1660. Find the area bounded by Pascal's limaçon

$$r = 2 + \cos\varphi.$$

1661. Find the area bounded by the parabola $r = a\sec^2\frac{\varphi}{2}$ and the two half-lines $\varphi = \frac{\pi}{4}$ and $\varphi = \frac{\pi}{2}$.

1662. Find the area of the ellipse $r = \frac{p}{1+\varepsilon\cos\varphi}$ $(\varepsilon < 1)$.

1663. Find the area bounded by the curve $r = 2a\cos 3\varphi$ and lying outside the circle $r = a$.

1664*. Find the area bounded by the curve $x^4 + y^4 = x^2 + y^2$.

Sec. 8. The Arc Length of a Curve

1°. The arc length in rectangular coordinates. The arc length s of a curve $y = f(x)$ contained between two points with abscissas $x = a$ and $x = b$ is

$$s = \int_a^b \sqrt{1+y'^2}\,dx.$$

Example 1. Find the length of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ (Fig. 49).

Solution. Differentiating the equation of the astroid, we get

$$y' = -\frac{y^{1/3}}{x^{1/3}}.$$

For this reason, we have for the arc length of a quarter of the astroid:

$$\frac{1}{4}s=\int_0^a \sqrt{1+\frac{y^{2/3}}{x^{2/3}}}\,dx=\int_0^a \frac{a^{1/3}}{x^{1/3}}\,dx=\frac{3}{2}a.$$

Whence $s=6a$.

2°. The arc length of a curve represented parametrically. If a curve is represented by equations in parametric form, $x=\varphi(t)$ and $y=\psi(t)$, then the arc length s of the curve is

$$s=\int_{t_1}^{t_2} \sqrt{x'^2+y'^2}\,dt,$$

where t_1 and t_2 are values of the parameter that correspond to the extremities of the arc.

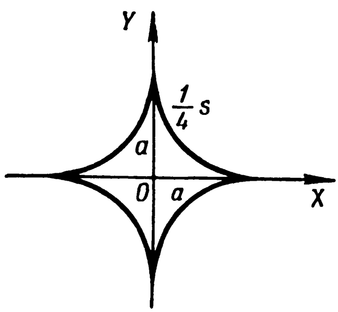

Fig 49

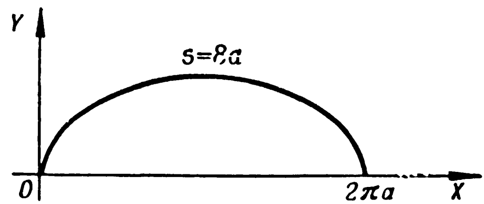

Fig. 50

Example 2. Find the length of one arc of the cycloid (Fig. 50)

$$\begin{cases} x=a(t-\sin t), \\ y=a(1-\cos t). \end{cases}$$

Solution. We have $\frac{dx}{dt}=a(1-\cos t)$ and $\frac{dy}{dt}=a\sin t$. Therefore,

$$s=\int_0^{2\pi} \sqrt{a^2(1-\cos t)^2+a^2\sin^2 t}\,dt=2a\int_0^{2\pi} \sin\frac{t}{2}\,dt=8a.$$

The limits of integration $t_1=0$ and $t_2=2\pi$ correspond to the extreme points of the arc of the cycloid.

If a curve is defined by the equation $r=f(\varphi)$ in polar coordinates, then the arc length s is

$$s=\int_\alpha^\beta \sqrt{r^2+r'^2}\,d\varphi,$$

where α and β are the values of the polar angle at the extreme points of the arc.

Example 3. Find the length of the entire curve $r=a\sin^3\frac{\varphi}{3}$ (Fig. 51). The entire curve is described by a point as φ ranges from 0 to 3π.

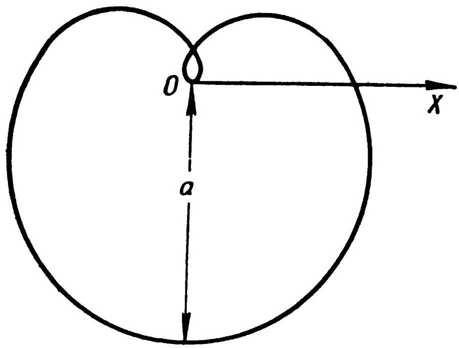

Fig. 51

Solution. We have $r'=a\sin^2\frac{\varphi}{3}\cos\frac{\varphi}{3}$, therefore the entire arc length of the curve is

$$s=\int_0^{3\pi}\sqrt{a^2\sin^6\frac{\varphi}{3}+a^2\sin^4\frac{\varphi}{3}\cos^2\frac{\varphi}{3}}\,d\varphi=a\int_0^{3\pi}\sin^2\frac{\varphi}{3}\,d\varphi=\frac{3\pi a}{2}.$$

1665. Compute the arc length of the semicubical parabola $y^2=x^3$ from the coordinate origin to the point $x=4$.

1666*. Find the length of the catenary $y=a\cosh\frac{x}{a}$ from the vertex $A(0,a)$ to the point $B(b,h)$.

1667. Compute the arc length of the parabola $y=2\sqrt{x}$ from $x=0$ to $x=1$.

1668. Find the arc length of the curve $y=e^x$ lying between the points $(0,1)$ and $(1,e)$.

1669. Find the arc length of the curve $y=\ln x$ from $x=\sqrt{3}$ to $x=\sqrt{8}$.

1670. Find the arc length of the curve $y=\arcsin(e^{-x})$ from $x=0$ to $x=1$.

1671. Compute the arc length of the curve $x=\ln\sec y$, lying between $y=0$ and $y=\frac{\pi}{3}$.

1672. Find the arc length of the curve $x=\frac{1}{4}y^2-\frac{1}{2}\ln y$ from $y=1$ to $y=e$.

1673. Find the length of the right branch of the tractrix

$$x=\sqrt{a^2-y^2}+a\ln\left|\frac{a+\sqrt{a^2-y^2}}{y}\right| \text{ from } y=a \text{ to } y=b\ (0<b<a).$$

1674. Find the length of the closed part of the curve $9ay^2=x(x-3a)^2$.

1675. Find the length of the curve $y=\ln\left(\coth\frac{x}{a}\right)$ from $x=a$ to $x=b$ $(0<a<b)$.

1676*. Find the arc length of the involute of the circle

$$\left.\begin{array}{l} x=a(\cos t+t\sin t), \\ y=a(\sin t-t\cos t) \end{array}\right\} \text{ from } t=0 \text{ to } t=T.$$

1677. Find the length of the evolute of the ellipse

$$x=\frac{c^2}{a}\cos^3 t;\quad y=\frac{c^2}{b}\sin^3 t\quad (c^2=a^2-b^2).$$

1678. Find the length of the curve

$$\left.\begin{array}{l} x=a(2\cos t-\cos 2t), \\ y=a(2\sin t-\sin 2t). \end{array}\right\}$$

1679. Find the length of the first turn of Archimedes' spiral $r=a\varphi$.

1680. Find the entire length of the cardioid $r=a(1+\cos\varphi)$.

1681. Find the arc length of that part of the parabola $r=a\sec^2\frac{\varphi}{2}$ which is cut off by a vertical line passing through the pole.

1682. Find the length of the hyperbolic spiral $r\varphi=1$ from the point $(2, 1/2)$ to the point $(1/2, 2)$.

1683. Find the arc length of the logarithmic spiral $r=ae^{m\varphi}$, lying inside the circle $r=a$.

1684. Find the arc length of the curve $\varphi=\frac{1}{2}\left(r+\frac{1}{r}\right)$ from $r=1$ to $r=3$.

Sec. 9. Volumes of Solids

1°. The volume of a solid of revolution. The volumes of solids formed by the revolution of a curvilinear trapezoid [bounded by the curve $y=f(x)$, the x-axis and two vertical lines $x=a$ and $x=b$] about the x- and y-axes are

expressed, respectively, by the formulas:

$$1)\ V_X=\pi\int_a^b y^2\,dx;\quad 2)\ V_Y=2\pi\int_a^b xy\,dx\ ^{*)}.$$

Example 1. Compute the volumes of solids formed by the revolution of a figure bounded by a single lobe of the sinusoidal curve $y=\sin x$ and by the segment $0\leqslant x\leqslant\pi$ of the x-axis about: a) the x-axis and b) the y-axis.

Solution.

$$a)\ V_X=\pi\int_0^\pi \sin^2 x\,dx=\frac{\pi^2}{2};$$

$$b)\ V_Y=2\pi\int_0^\pi x\sin x\,dx=2\pi\left(-x\cos x+\sin x\right)_0^\pi=2\pi^2.$$

The volume of a solid formed by revolution about the y-axis of a figure bounded by the curve $x=g(y)$, the y-axis and by two parallel lines $y=c$ and $y=d$, may be determined from the formula

$$V_Y=\pi\int_c^d x^2\,dy,$$

obtained from formula (1), given above, by interchanging the coordinates x and y.

If the curve is defined in a different form (parametrically, in polar coordinates, etc.), then in the foregoing formulas we must change the variable of integration in appropriate fashion.

In the more general case, the volumes of solids formed by the revolution about the x- and y-axes of a figure bounded by the curves $y_1=f_1(x)$ and $y_2=f_2(x)$ [where $f_1(x)\leqslant f_2(x)$], and the straight lines $x=a$ and $x=b$ are, respectively, equal to

$$V_X=\pi\int_a^b (y_2^2-y_1^2)\,dx$$

and

$$V_Y=2\pi\int_a^b x(y_2-y_1)\,dx.$$

Example 2. Find the volume of a torus formed by the rotation of the circle $x^2+(y-b)^2=a^2$ $(b\geqslant a)$ about the x-axis (Fig. 52).

---

*) The solid is formed by the revolution, about the y-axis, of a curvilinear trapezoid bounded by the curve $y=f(x)$ and the straight lines $x=a$, $x=b$, and $y=0$. For a volume element we take the volume of that part of the solid formed by revolving about the y-axis a rectangle with sides y and dx at a distance x from the y-axis. Then the volume element $dV_Y=2\pi xy\,dx$, whence $V_Y=2\pi\int_a^b xy\,dx$.

Solution. We have

$$y_1=b-\sqrt{a^2-x^2} \text{ and } y_2=b+\sqrt{a^2-x^2}.$$

Therefore,

$$V_X=\pi\int_{-a}^{a}[(b+\sqrt{a^2-x^2})^2-(b-\sqrt{a^2-x^2})^2]\,dx=$$

$$=4\pi b\int_{-a}^{a}\sqrt{a^2-x^2}\,dx=2\pi^2a^2b$$

(the latter integral is taken by the substitution $x=a\sin t$).

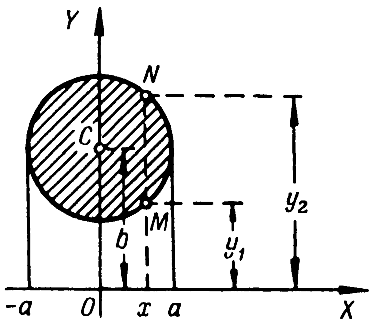

Fig 52

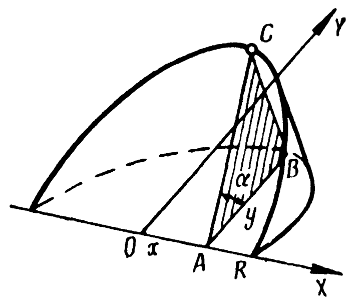

Fig. 53

The volume of a solid obtained by the rotation, about the polar axis, of a sector formed by an arc of the curve $r=F(\varphi)$ and by two radius vectors $\varphi=\alpha$, $\varphi=\beta$ may be computed from the formula

$$V_P=\frac{2}{3}\pi\int_{\alpha}^{\beta}r^3\sin\varphi\,d\varphi.$$

This same formula is conveniently used when seeking the volume obtained by the rotation, about the polar axis, of some closed curve defined in polar coordinates.

Example 3. Determine the volume formed by the rotation of the curve $r=a\sin 2\varphi$ about the polar axis.

Solution.

$$V_P=2\cdot\frac{2}{3}\pi\int_{0}^{\frac{\pi}{2}}r^3\sin\varphi\,d\varphi=\frac{4}{3}\pi a^3\int_{0}^{\frac{\pi}{2}}\sin^3 2\varphi\sin\varphi\,d\varphi=$$

$$=\frac{32}{3}\pi a^3\int_{0}^{\frac{\pi}{2}}\sin^4\varphi\cos^3\varphi\,d\varphi=\frac{64}{105}\pi a^3.$$

2°. Computing the volumes of solids from known cross-sections. If $S=S(x)$ is the cross-sectional area cut off by a plane perpendicular to some straight line (which we take to be the x-axis) at a point with abscissa x, then the volume of the solid is

$$V=\int_{x_1}^{x_2} S(x)\,dx,$$

where x_1 and x_2 are the abscissas of the extreme cross-sections of the solid.

Example 4. Determine the volume of a wedge cut off a circular cylinder by a plane passing through the diameter of the base and inclined to the base at an angle α. The radius of the base is R (Fig. 53).

Solution. For the x-axis we take the diameter of the base along which the cutting plane intersects the base, and for the y-axis we take the diameter of the base perpendicular to it. The equation of the circumference of the base is $x^2+y^2=R^2$.

The area of the section ABC at a distance x from the origin O is $S(x)=\text{area}\ \triangle ABC=\frac{1}{2}AB\cdot BC=\frac{1}{2}yy\tan\alpha=\frac{y^2}{2}\tan\alpha$. Therefore, the sought-for volume of the wedge is

$$V=2\cdot\frac{1}{2}\int_0^R y^2\tan\alpha\,dx=\tan\alpha\int_0^R (R^2-x^2)\,dx=\frac{2}{3}\tan\alpha\,R^3.$$

1685. Find the volume of a solid formed by rotation, about the x-axis, of an area bounded by the x-axis and the parabola $y=ax-x^2$ $(a>0)$.

1686. Find the volume of an ellipsoid formed by the rotation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about the x-axis.

1687. Find the volume of a solid formed by the rotation, about the x-axis, of an area bounded by the catenary $y=a\cosh\frac{x}{a}$, the x-axis, and the straight lines $x=\pm a$.

1688. Find the volume of a solid formed by the rotation, about the x-axis, of the curve $y=\sin^2 x$ in the interval between $x=0$ and $x=\pi$.

1689. Find the volume of a solid formed by the rotation, about the x-axis, of an area bounded by the semicubical parabola $y^2=x^3$, the x-axis, and the straight line $x=1$.

1690. Find the volume of a solid formed by the rotation of the same area (as in Problem 1689) about the y-axis.

1691. Find the volumes of the solids formed by the rotation of an area bounded by the lines $y=e^x$, $x=0$, $y=0$ about: a) the x-axis and b) the y-axis.

1692. Find the volume of a solid formed by the rotation, about the y-axis, of that part of the parabola $y^2=4ax$ which is cut off by the straight line $x=a$.

1693. Find the volume of a solid formed by the rotation, about the straight line $x=a$, of that part of the parabola $y^2=4ax$ which is cut off by this line.

1694. Find the volume of a solid formed by the rotation, about the straight line $y=-p$, of a figure bounded by the parabola $y^2=2px$ and the straight line $x=\frac{p}{2}$.

1695. Find the volume of a solid formed by the rotation, about the x-axis, of the area contained between the parabolas $y=x^2$ and $y=\sqrt{x}$.

1696. Find the volume of a solid formed by the rotation, about the x-axis, of a loop of the curve $(x-4a)y^2=ax(x-3a)$ $(a>0)$.

1697. Find the volume of a solid generated by the rotation of the cyssoid $y^2=\frac{x^3}{2a-x}$ about its asymptote $x=2a$.

1698. Find the volume of a paraboloid of revolution whose base has radius R and whose altitude is H.

1699. A right parabolic segment whose base is $2a$ and altitude h is in rotation about the base. Determine the volume of the resulting solid of revolution (Cavalieri's "lemon").

1700. Show that the volume of a part cut by the plane $x=2a$ off a solid formed by the rotation of the equilateral hyperbola $x^2-y^2=a^2$ about the x-axis is equal to the volume of a sphere of radius a.

1701. Find the volume of a solid formed by the rotation of a figure bounded by one arc of the cycloid $x=a(t-\sin t)$, $y=a(1-\cos t)$ and the x-axis, about: a) the x-axis, b) the y-axis, and c) the axis of symmetry of the figure.

1702. Find the volume of a solid formed by the rotation of the astroid $x=a\cos^3 t$, $y=b\sin^3 t$ about the y-axis.

1703. Find the volume of a solid obtained by rotating the cardioid $r=a(1+\cos\varphi)$ about the polar axis.

1704. Find the volume of a solid formed by rotation of the curve $r=a\cos^2\varphi$ about the polar axis.

1705. Find the volume of an obelisk whose parallel bases are rectangles with sides A, B and a, b, and the altitude is h.

1706. Find the volume of a right elliptic cone whose base is an ellipse with semi-axes a and b, and altitude h.

1707. On the chords of the astroid $x^{2/3}+y^{2/3}=a^{2/3}$, which are parallel to the x-axis, are constructed squares whose sides are equal to the lengths of the chords and whose planes are perpendicular to the xy-plane. Find the volume of the solid formed by these squares.

1708. A circle undergoing deformation is moving so that one of the points of its circumference lies on the y-axis, the centre describes an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and the plane of the circle is perpendicular to the xy-plane. Find the volume of the solid generated by the circle.

1709. The plane of a moving triangle remains perpendicular to the stationary diameter of a circle of radius a. The base of the triangle is a chord of the circle, while its vertex slides along a straight line parallel to the stationary diameter at a distance h from the plane of the circle. Find the volume of the solid (called a conoid) formed by the motion of this triangle from one end of the diameter to the other.

1710. Find the volume of the solid bounded by the cylinders $x^2+z^2=a^2$ and $y^2+z^2=a^2$.

1711. Find the volume of the segment cut off from the elliptic paraboloid $\frac{y^2}{2p}+\frac{z^2}{2q}=x$ by the plane $x=a$.

1712. Find the volume of the solid bounded by the hyperboloid of one sheet $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ and the planes $z=0$ and $z=h$.

1713. Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Sec. 10. The Area of a Surface of Revolution

The area of a surface formed by the rotation, about the x-axis, of an arc of the curve $y=f(x)$ between the points $x=a$ and $x=b$, is expressed by the formula

$$S_X=2\pi\int_a^b y\frac{ds}{dx}\,dx=2\pi\int_a^b y\sqrt{1+y'^2}\,dx \tag{1}$$

(ds is the differential of the arc of the curve).

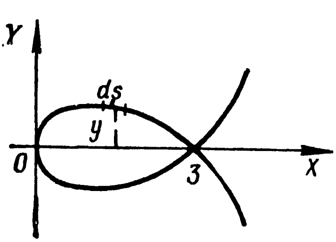

Fig. 54

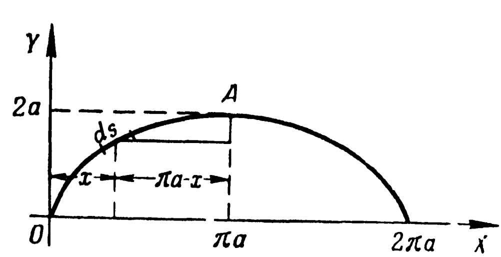

Fig. 55

If the equation of the curve is represented differently, the area of the surface S_X is obtained from formula (1) by an appropriate change of variables.

Example 1. Find the area of a surface formed by rotation, about the x-axis, of a loop of the curve $9y^2=x(3-x)^2$ (Fig. 54).

Solution. For the upper part of the curve, when $0 \leqslant x \leqslant 3$, we have $y=\frac{1}{3}(3-x)\sqrt{x}$. Whence the differential of the arc $ds=\frac{x+1}{2\sqrt{x}}dx$. From formula (1) the area of the surface

$$S=2\pi\int_0^3 \frac{1}{3}(3-x)\sqrt{x}\,\frac{x+1}{2\sqrt{x}}\,dx=3\pi.$$

Example 2. Find the area of a surface formed by the rotation of one arc of the cycloid $x=a(t-\sin t)$; $y=a(1-\cos t)$ about its axis of symmetry (Fig. 55).

Solution. The desired surface is formed by rotation of the arc OA about the straight line AB, the equation of which is $x=\pi a$. Taking y as the independent variable and noting that the axis of rotation AB is displaced relative to the y-axis a distance πa, we will have

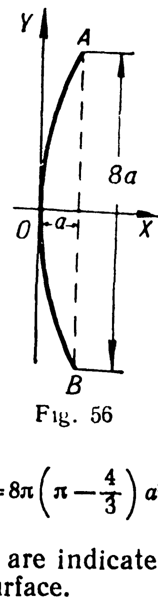

Fig. 56

$$S=2\pi\int_0^{2a}(\pi a-x)\frac{ds}{dy}\cdot dy.$$

Passing to the variable t, we obtain

$$S=2\pi\int_0^{\pi}(\pi a-at+a\sin t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=$$

$$=2\pi\int_0^{\pi}(\pi a-at+a\sin t)\,2a\sin\frac{t}{2}\,dt=$$

$$=4\pi a^2\int_0^{\pi}\left(\pi\sin\frac{t}{2}-t\sin\frac{t}{2}+\sin t\sin\frac{t}{2}\right)dt=$$

$$=4\pi a^2\left[-2\pi\cos\frac{t}{2}+2t\cos\frac{t}{2}-4\sin\frac{t}{2}+\frac{4}{3}\sin^3\frac{t}{2}\right]_0^{\pi}=8\pi\left(\pi-\frac{4}{3}\right)a^2.$$

1714. The dimensions of a parabolic mirror AOB are indicated in Fig. 56. It is required to find the area of its surface.

1715. Find the area of the surface of a spindle obtained by rotation of a lobe of the sinusoidal curve $y=\sin x$ about the x-axis.

1716. Find the area of the surface formed by the rotation of a part of the tangential curve $y=\tan x$ from $x=0$ to $x=\frac{\pi}{4}$, about the x-axis.

1717. Find the area of the surface formed by rotation, about the x-axis, of an arc of the curve $y=e^{-x}$, from $x=0$ to $x=+\infty$.

1718. Find the area of the surface (called a *catenoid*) formed by the rotation of a catenary $y=a\cosh\frac{x}{a}$ about the x-axis from $x=0$ to $x=a$.

1719. Find the area of the surface of rotation of the astroid $x^{2/3}+y^{2/3}=a^{2/3}$ about the y-axis.

1720. Find the area of the surface of rotation of the curve $x=\frac{1}{4}y^2-\frac{1}{2}\ln y$ about the x-axis from $y=1$ to $y=e$.

1721*. Find the surface of a torus formed by rotation of the circle $x^2+(y-b)^2=a^2$ about the x-axis $(b>a)$.

1722. Find the area of the surface formed by rotation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about: 1) the x-axis, 2) the y-axis $(a>b)$.

1723. Find the area of the surface formed by rotation of one arc of the cycloid $x=a(t-\sin t)$ and $y=a(1-\cos t)$ about: a) the x-axis, b) the y-axis, c) the tangent to the cycloid at its highest point.

1724. Find the area of the surface formed by rotation, about the x-axis, of the cardioid

$$\left.\begin{aligned} x&=a(2\cos t-\cos 2t),\\ y&=a(2\sin t-\sin 2t).\end{aligned}\right\}$$

1725. Determine the area of the surface formed by the rotation of the lemniscate $r^2=a^2\cos 2\varphi$ about the polar axis.

1726. Determine the area of the surface formed by the rotation of the cardioid $r=2a(1+\cos\varphi)$ about the polar axis.

Sec. 11. Moments. Centres of Gravity. Guldin's Theorems

1°. Static moment. The *static moment* relative to the l-axis of a material point A having mass m and at a distance d from the l-axis is the quantity $M_l=md$.

The *static moment* relative to the l-axis of a system of n material points with masses $m_1, m_2, \ldots, m_n$ lying in the plane of the axis and at distances $d_1, d_2, \ldots, d_n$ is the sum

$$M_l=\sum_{i=1}^{n} m_i d_i, \tag{1}$$

where the distances of points lying on one side of the l-axis have the plus sign, those on the other side have the minus sign. In a similar manner we define the *static moment of a system of points* relative to a plane.

If the masses continuously fill the line or figure of the xy-plane, then the static moments M_X and M_Y about the x- and y-axes are expressed (respectively) as integrals and not as the sums (1). For the cases of geometric figures, the density is considered equal to unity.

In particular: 1) for the curve $x=x(s)$; $y=y(s)$, where the parameter s is the arc length, we have

$$M_X=\int_0^L y(s)\,ds;\quad M_Y=\int_0^L x(s)\,ds \tag{2}$$

($ds=\sqrt{(dx)^2+(dy)^2}$ is the differential of the arc);

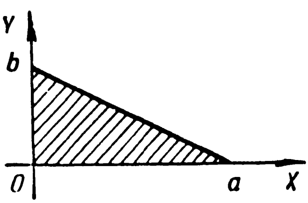

Fig. 57

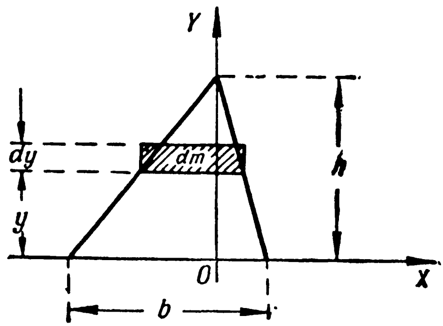

Fig. 58

2) for a plane figure bounded by the curve $y=y(x)$, the x-axis and two vertical lines $x=a$ and $y=b$, we obtain

$$M_X=\frac{1}{2}\int_a^b y\,|\,y\,|\,dx;\quad M_Y=\int_a^b x\,|\,y\,|\,dx. \tag{3}$$

Example 1. Find the static moments about the x- and y-axes of a triangle bounded by the straight lines: $\frac{x}{a}+\frac{y}{b}=1$, $x=0$, $y=0$ (Fig. 57)

Solution. Here, $y=b\left(1-\frac{x}{a}\right)$. Applying formula (3), we obtain

$$M_X=\frac{b^2}{2}\int_0^a\left(1-\frac{x}{a}\right)^2dx=\frac{ab^2}{6}$$

and

$$M_Y=b\int_0^a x\left(1-\frac{x}{a}\right)dx=\frac{a^2b}{6}.$$

2°. **Moment of inertia.** The *moment of inertia*, about an l-axis, of a material point of mass m at a distance d from the l-axis, is the number $I_l=md^2$.

The *moment of inertia*, about an l-axis, of a system of n material points with masses $m_1, m_2, \ldots, m_n$ is the sum

$$I_l=\sum_{i=1}^{n} m_i d_i^2,$$

where $d_1, d_2 \dots, d_n$ are the distances of the points from the l-axis. In the case of a continuous mass, we get an appropriate integral in place of a sum.

Example 2. Find the moment of inertia of a triangle with base b and altitude h about its base.

Solution. For the base of the triangle we take the x-axis, for its altitude, the y-axis (Fig 58).

Divide the triangle into infinitely narrow horizontal strips of width dy, which play the role of elementary masses dm. Utilizing the similarity of triangles, we obtain

$$dm = b\frac{h-y}{h}\,dy$$

and

$$dI_X = y^2 dm = \frac{b}{h}\,y^2(h-y)\,dy.$$

Whence

$$I_X = \frac{b}{h}\int_0^h y^2(h-y)\,dy = \frac{1}{12}bh^3.$$

3°. Centre of gravity. The *coordinates of the centre of gravity* of a plane figure (arc or area) of mass M are computed from the formulas

$$\bar{x} = \frac{M_Y}{M}, \quad \bar{y} = \frac{M_X}{M},$$

where M_X and M_Y are the static moments of the mass. In the case of geometric figures, the mass M is numerically equal to the corresponding arc or area.

For the coordinates of the centre of gravity $(\bar{x}, \bar{y})$ of an arc of the plane curve $y=f(x)$ $(a \leqslant x \leqslant b)$, connecting the points $A[a, f(a)]$ and $B[b, f(b)]$, we have

$$\bar{x} = \frac{\int_A^B x\,ds}{s} = \frac{\int_a^b x\sqrt{1+(y')^2}\,dx}{\int_a^b \sqrt{1+(y')^2}\,dx}, \qquad \bar{y} = \frac{\int_A^B y\,ds}{s} = \frac{\int_a^b y\sqrt{1+(y')^2}\,dx}{\int_a^b \sqrt{1+(y')^2}\,dx}.$$

The coordinates of the centre of gravity $(\bar{x}, \bar{y})$ of the curvilinear trapezoid $a \leqslant x \leqslant b$, $0 \leqslant y \leqslant f(x)$ may be computed from the formulas

$$\bar{x} = \frac{\int_a^b xy\,dx}{S}, \qquad \bar{y} = \frac{\frac{1}{2}\int_a^b y^2\,dx}{S},$$

where $S = \int_a^b y\,dx$ is the area of the figure.

There are similar formulas for the coordinates of the centre of gravity of a volume.

Example 3. Find the centre of gravity of an arc of the semicircle $x^2+y^2=a^2$; $(y \geqslant 0)$ (Fig. 59).

Solution. We have

$$y=\sqrt{a^2-x^2};\quad y'=\frac{-x}{\sqrt{a^2-x^2}}$$

and

$$ds=\sqrt{1+(y')^2}\,dx=\frac{a\,dx}{\sqrt{a^2-x^2}}.$$

Whence

$$M_Y=\int_{-a}^{a}x\,ds=\int_{-a}^{a}\frac{ax}{\sqrt{a^2-x^2}}\,dx=0,$$

$$M_X=\int_{-a}^{a}y\,ds=\int_{-a}^{a}\sqrt{a^2-x^2}\,\frac{a\,dx}{\sqrt{a^2-x^2}}=2a^2,$$

$$M=\int_{-a}^{a}\frac{a\,dx}{\sqrt{a^2-x^2}}=\pi a.$$

Hence,

$$\bar{x}=0;\quad \bar{y}=\frac{2}{\pi}a.$$

4°. Guldin's theorems.

Theorem 1. The area of a surface obtained by the rotation of an arc of a plane curve about some axis lying in the same plane as the curve and not intersecting it is equal to the product of the length of the curve by the circumference of the circle described by the centre of gravity of the arc of the curve.

Theorem 2. The volume of a solid obtained by rotation of a plane figure about some axis lying in the plane of the figure and not intersecting it is equal to the product of the area of this figure by the circumference of the circle described by the centre of gravity of the figure.

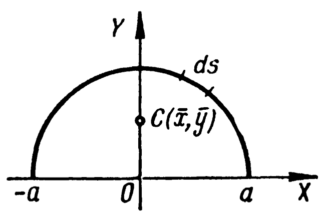

Fig. 59

1727. Find the static moments about the coordinate axes of a segment of the straight line

$$\frac{x}{a}+\frac{y}{b}=1,$$

lying between the axes.

1728. Find the static moments of a rectangle, with sides a and b, about its sides.

1729. Find the static moments, about the x- and y-axes, and the coordinates of the centre of gravity of a triangle bounded by the straight lines $x+y=a$, $x=0$, and $y=0$.

1730. Find the static moments, about the x- and y-axes, and the coordinates of the centre of gravity of an arc of the astroid

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}},$$

lying in the first quadrant.

1731. Find the static moment of the circle

$$r=2a\sin\varphi$$

about the polar axis.

1732. Find the coordinates of the centre of gravity of an arc of the catenary

$$y=a\cosh\frac{x}{a}$$

from $x=-a$ to $x=a$.

1733. Find the centre of gravity of an arc of a circle of radius a subtending an angle 2α.

1734. Find the coordinates of the centre of gravity of the arc of one arch of the cycloid

$$x=a(t-\sin t);\ \ y=a(1-\cos t).$$

1735. Find the coordinates of the centre of gravity of an area bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the coordinate axes ($x\geqslant 0$, $y\geqslant 0$).

1736. Find the coordinates of the centre of gravity of an area bounded by the curves

$$y=x^2,\ \ y=\sqrt{x}.$$

1737. Find the coordinates of the centre of gravity of an area bounded by the first arch of the cycloid

$$x=a(t-\sin t),\ \ y=a(1-\cos t)$$

and the x-axis.

1738.** Find the centre of gravity of a hemisphere of radius a lying above the xy-plane with centre at the origin.

1739.** Find the centre of gravity of a homogeneous right circular cone with base radius r and altitude h.

1740.** Find the centre of gravity of a homogeneous hemisphere of radius a lying above the xy-plane with centre at the origin.

1741. Find the moment of inertia of a circle of radius a about its diameter.

1742. Find the moments of inertia of a rectangle with sides a and b about its sides.

1743. Find the moment of inertia of a right parabolic segment with base $2b$ and altitude h about its axis of symmetry.

1744. Find the moments of inertia of the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about its principal axes.

1745.** Find the polar moment of inertia of a circular ring with radii R_1 and R_2 $(R_1<R_2)$, that is, the moment of inertia about the axis passing through the centre of the ring and perpendicular to its plane.

1746.** Find the moment of inertia of a homogeneous right circular cone with base radius R and altitude H about its axis.

1747.** Find the moment of inertia of a homogeneous sphere of radius a and of mass M about its diameter.

1748. Find the surface and volume of a torus obtained by rotating a circle of radius a about an axis lying in its plane and at a distance b $(b>a)$ from its centre.

1749. a) Determine the position of the centre of gravity of an arc of the astroid $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ lying in the first quadrant.

b) Find the centre of gravity of an area bounded by the curves $y^2=2px$ and $x^2=2py$.

1750.** a) Find the centre of gravity of a semicircle using Guldin's theorem.

b) Prove by Guldin's theorem that the centre of gravity of a triangle is distant from its base by one third of its altitude

Sec. 12. Applying Definite Integrals to the Solution of Physical Problems

1°. The path traversed by a point. If a point is in motion along some curve and the absolute value of the velocity $v=f(t)$ is a known function of the time t, then the *path* traversed by the point in an interval of time $[t_1, t_2]$ is

$$s=\int_{t_1}^{t_2} f(t)\,dt.$$

Example 1. The velocity of a point is

$$v=0.1t^3 \text{ m/sec.}$$

Find the path s covered by the point in the interval of time $T=10$ sec following the commencement of motion. What is the mean velocity of motion during this interval?

Solution. We have:

$$s=\int_{0}^{10} 0.1t^3dt=0.1\left.\frac{t^4}{4}\right|_0^{10}=250 \text{ metres}$$

and

$$v_{\text{mean}}=\frac{s}{T}=25 \text{ m/sec.}$$

2°. **The work of a force.** If a variable force $X=f(x)$ acts in the direction of the x-axis, then the *work of this force* over an interval $[x_1, x_2]$ is

$$A=\int_{x_1}^{x_2} f(x)\,dx.$$

Example 2. What work has to be performed to stretch a spring 6 cm, if a force of 1 kgf stretches it by 1 cm?

Solution. According to Hook's law the force X kgf stretching the spring by x_m is equal to $X=kx$, where k is a proportionality constant.

Putting $x=0.01$ m and $X=1$ kgf, we get $k=100$ and, hence, $X=100x$. Whence the sought-for work is

$$A=\int_{0}^{0.06} 100\,x\,dx=50\,x^2\Big|_0^{0.06}=0.18 \text{ kgm}$$

3°. **Kinetic energy.** The *kinetic energy* of a material point of mass m and velocity v is defined as

$$K=\frac{mv^2}{2}.$$

The *kinetic energy* of a system of n material points with masses $m_1, m_2, \ldots, m_n$ having respective velocities $v_1, v_2, \ldots, v_n$, is equal to

$$K=\sum_{i=1}^{n}\frac{m_iv_i^2}{2}. \tag{1}$$

To compute the kinetic energy of a solid, the latter is appropriately partitioned into elementary particles (which play the part of material points); then by summing the kinetic energies of these particles we get, in the limit, an integral in place of the sum (1).

Example 3. Find the kinetic energy of a homogeneous circular cylinder of density δ with base radius R and altitude h rotating about its axis with angular velocity ω.

Solution. For the elementary mass dm we take the mass of a hollow cylinder of altitude h with inner radius r and wall thickness dr (Fig. 60). We have:

$$dm=2\pi r\cdot h\delta\,dr.$$

Since the linear velocity of the mass dm is equal to $v=r\omega$, the elementary kinetic energy is

$$dK=\frac{v^2dm}{2}=\pi r^3\omega^2h\delta\,dr.$$

Whence

$$K=\pi\omega^2 h\delta\int_0^R r^3dr=\frac{\pi\omega^2\delta R^4 h}{4}.$$

4°. **Pressure of a liquid.** To compute the force of *liquid pressure* we use Pascal's law, which states that the force of pressure of a liquid on an area S at a depth of immersion h is

$$p=\gamma hS,$$

where γ is the specific weight of the liquid.

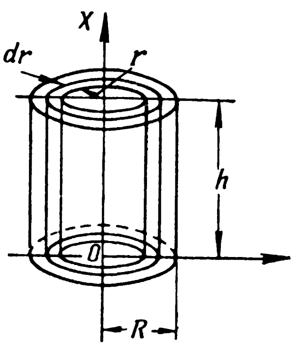

Fig. 60

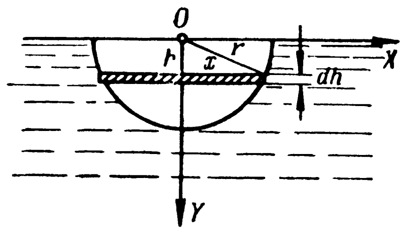

Fig 61

Example 4. Find the force of pressure experienced by a semicircle of radius r submerged vertically in water so that its diameter is flush with the water surface (Fig 61).

Solution. We partition the area of the semicircle into elements—strips parallel to the surface of the water. The area of one such element (ignoring higher-order infinitesimals) located at a distance h from the surface is

$$ds=2xdh=2\sqrt{r^2-h^2}\,dh.$$

The pressure experienced by this element is

$$dP=\gamma h\,ds=2\gamma h\sqrt{r^2-h^2}\,dh,$$

where γ is the specific weight of the water equal to unity.

Whence the entire pressure is

$$P=2\int_0^r h\sqrt{r^2-h^2}\,dh=-\frac{2}{3}(r^2-h^2)^{\frac{3}{2}}\Big|_0^r=\frac{2}{3}r^3.$$

1751. The velocity of a body thrown vertically upwards with initial velocity v_0 (air resistance neglected), is given by the

formula

$$v = v_0 - gt,$$

where t is the time that elapses and g is the acceleration of gravity. At what distance from the initial position will the body be in t seconds from the time it is thrown?

1752. The velocity of a body thrown vertically upwards with initial velocity v_0 (air resistance allowed for) is given by the formula

$$v = c \cdot \tan\left(-\frac{g}{c}t + \arctan\frac{v_0}{c}\right),$$

where t is the time, g is the acceleration of gravity, and c is a constant. Find the altitude reached by the body.

1753. A point on the x-axis performs harmonic oscillations about the coordinate origin; its velocity is given by the formula

$$v = v_0 \cos \omega t,$$

where t is the time and v_0, ω are constants.

Find the law of oscillation of a point if when $t = 0$ it had an abscissa $x = 0$. What is the mean value of the absolute magnitude of the velocity of the point during one cycle?

1754. The velocity of motion of a point is $v = te^{-0.01t}$ m/sec. Find the path covered by the point from the commencement of motion to full stop.

1755. A rocket rises vertically upwards. Considering that when the rocket thrust is constant, the acceleration due to decreasing weight of the rocket increases by the law $j = \frac{A}{a - bt}$ $(a - bt > 0)$, find the velocity at any instant of time t, if the initial velocity is zero. Find the altitude reached at time $t = t_1$.

1756*. Calculate the work that has to be done to pump the water out of a vertical cylindrical barrel with base radius R and altitude H.

1757. Calculate the work that has to be done in order to pump the water out of a conical vessel with vertex downwards, the radius of the base of which is R and the altitude H.

1758. Calculate the work to be done in order to pump water out of a semispherical boiler of radius $R = 10$ m.

1759. Calculate the work needed to pump oil out of a tank through an upper opening (the tank has the shape of a cylinder with horizontal axis) if the specific weight of the oil is γ, the length of the tank H and the radius of the base R.

1760.** What work has to be done to raise a body of mass m from the earth's surface (radius R) to an altitude h? What is the work if the body is removed to infinity?

1761.** Two electric charges $e_0 = 100$ CGSE and $e_1 = 200$ CGSE lie on the x-axis at points $x_0 = 0$ and $x_1 = 1$ cm, respectively. What work will be done if the second charge is moved to point $x_2 = 10$ cm?

1762.** A cylinder with a movable piston of diameter $D = 20$ cm and length $l = 80$ cm is filled with steam at a pressure $p = 10$ kgf cm^2. What work must be done to halve the volume of the steam with temperature kept constant (isothermic process)?

1763.** Determine the work performed in the adiabatic expansion of air (having initial volume $v_0 = 1$ m^3 and pressure $p_0 = 1$ kgf/cm^2) to volume $v_1 = 10$ m^3?

1764.** A vertical shaft of weight P and radius a rests on a bearing AB (Fig. 62). The frictional force between a small part σ of the base of the shaft and the surface of the support in contact with it is $F = \mu p \sigma$, where $p = \text{const}$ is the pressure of the shaft on the surface of the support referred to unit area of the support, while μ is the coefficient of friction. Find the work done by the frictional force during one revolution of the shaft.

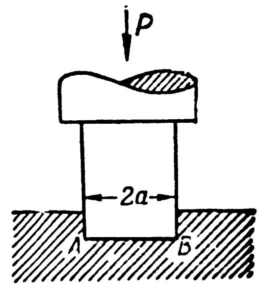

Fig. 62

1765.** Calculate the kinetic energy of a disk of mass M and radius R rotating with angular velocity ω about an axis that passes through its centre perpendicular to its plane.

1766. Calculate the kinetic energy of a right circular cone of mass M rotating with angular velocity ω about its axis, if the radius of the base of the cone is R and the altitude is H.

1767*. What work has to be done to stop an iron sphere of radius $R = 2$ metres rotating with angular velocity $\omega = 1{,}000$ rpm about its diameter? (Specific weight of iron, $\gamma = 7.8$ g/cm^3.)

1768. A vertical triangle with base b and altitude h is submerged vertex downwards in water so that its base is on the surface of the water. Find the pressure of the water.

1769. A vertical dam has the shape of a trapezoid. Calculate the water pressure on the dam if we know that the upper base $a = 70$ m, the lower base $b = 50$ m, and the height $h = 20$ m.

1770. Find the pressure of a liquid, whose specific weight is γ, on a vertical ellipse (with axes $2a$ and $2b$) whose centre is submerged in the liquid to a distance h, while the major axis $2a$ of the ellipse is parallel to the level of the liquid ($h \geqslant b$).

1771. Find the water pressure on a vertical circular cone with radius of base R and altitude H submerged in water vertex downwards so that its base is on the surface of the water.

Miscellaneous Problems

1772. Find the mass of a rod of length $l=100$ cm if the linear density of the rod at a distance x cm from one of its ends is

$$\delta=2+0.001\,x^2 \text{ g/cm}.$$

1773. According to empirical data the specific thermal capacity of water at a temperature $t°$ C ($0\leqslant t\leqslant 100°$) is

$$c=0.9983-5.184\times 10^{-5}\,t+6.912\times 10^{-7}\,t^2.$$

What quantity of heat has to be expended to heat 1 g of water from 0° C to 100° C?

1774. The wind exerts a uniform pressure p g/cm^2 on a door of width b cm and height h cm. Find the moment of the pressure of the wind striving to turn the door on its hinges.

1775. What is the force of attraction of a material rod of length l and mass M on a material point of mass m lying on a straight line with the rod at a distance a from one of its ends?

1776.** In the case of steady-state laminar flow of a liquid through a pipe of circular cross-section of radius a, the velocity of flow v at a point distant r from the axis of the pipe is given by the formula

$$v=\frac{p}{4\mu l}(a^2-r^2),$$

where p is the pressure difference at the ends of the pipe, μ is the coefficient of viscosity, and l is the length of the pipe. Determine the *discharge of liquid* Q (that is, the quantity of liquid flowing through a cross-section of the pipe in unit time).

1777*. The conditions are the same as in Problem 1776, but the pipe has a rectangular cross-section, and the base a is great compared with the altitude $2b$. Here the rate of flow v at a point $M(x,y)$ is defined by the formula

$$v=\frac{p}{2\mu l}[b^2-(b-y)^2].$$

Determine the discharge of liquid Q.

1778.** In studies of the dynamic qualities of an automobile, use is frequently made of special types of diagrams: the velocities v are laid off on the x-axis, and the reciprocals of corresponding accelerations a, on the y-axis. Show that the area S bounded by an arc of this graph, by two ordinates $v=v_1$ and $v=v_2$, and by the x-axis is numerically equal to the time needed to increase the velocity of motion of the automobile from v_1 to v_2 *(acceleration time).*

1779. A horizontal beam of length l is in equilibrium due to a downward vertical load uniformly distributed over the length of the beam, and of support reactions A and $B\left(A=B=\frac{Q}{2}\right)$, directed vertically upwards. Find the bending moment M_x in a cross-section x, that is, the moment about the point P with abscissa x of all forces acting on the portion of the beam AP.

1780. A horizontal beam of length l is in equilibrium due to support reactions A and B and a load distributed along the length of the beam with intensity $q=kx$, where x is the distance from the left support and k is a constant factor. Find the bending moment M_x in cross-section x.

Note. The intensity of load distribution is the load (force) referred to unit length.

1781*. Find the quantity of heat released by an alternating sinusoidal current

$$I=I_0\sin\left(\frac{2\pi}{T}t-\varphi\right)$$

during a cycle T in a conductor with resistance R.

Chapter VI

FUNCTIONS OF SEVERAL VARIABLES

Sec. 1. Basic Notions

1°. The concept of a function of several variables. Functional notation. A variable quantity z is called a single-valued function of two variables x, y, if to each set of their values (x, y) in a given range there corresponds a unique value of z. The variables x and y are called *arguments* or *independent variables*. The functional relation is denoted by

$$z = f(x, y).$$

Similarly, we define functions of three or more arguments.

Example 1. Express the volume of a cone V as a function of its generatrix x and of its base radius y.

Solution. From geometry we know that the volume of a cone is

$$V = \frac{1}{3}\pi y^2 h,$$

where h is the altitude of the cone. But $h = \sqrt{x^2 - y^2}$. Hence,

$$V = \frac{1}{3}\pi y^2 \sqrt{x^2 - y^2}.$$

This is the desired functional relation.

The value of the function $z = f(x, y)$ at a point $P(a, b)$, that is, when $x = a$ and $y = b$, is denoted by $f(a, b)$ or $f(P)$. Generally speaking, the geometric representation of a function like $z = f(x, y)$ in a rectangular coordinate system X, Y, Z is a surface (Fig. 63).

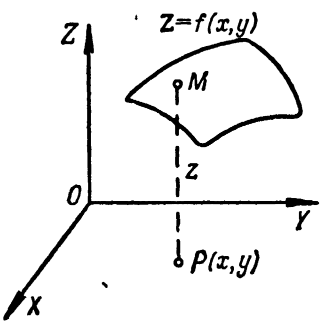

Fig. 63

Example 2. Find $f(2, -3)$ and $f\left(1, \frac{y}{x}\right)$ if

$$f(x, y) = \frac{x^2 + y^2}{2xy}.$$

Solution. Substituting $x = 2$ and $y = -3$, we find

$$f(2, -3) = \frac{2^2 + (-3)^2}{2 \cdot 2 \cdot (-3)} = -\frac{13}{12}.$$

Putting $x=1$ and replacing y by $\frac{y}{x}$, we will have

$$f\left(1, \frac{y}{x}\right)=\frac{1+\left(\frac{y}{x}\right)^2}{2\cdot 1\left(\frac{y}{x}\right)}=\frac{x^2+y^2}{2xy},$$

that is, $f\left(1, \frac{y}{x}\right)=f(x, y)$.

2°. Domain of definition of a function. By the *domain of definition* of a function $z=f(x, y)$ we understand a set of points (x, y) in an xy-plane in which the given function is defined (that is to say, in which it takes on definite real values) In the simplest cases, the domain of definition of a function is a finite or infinite part of the xy-plane bounded by one or several curves (the *boundary of the domain*).

Similarly, for a function of three variables $u=f(x, y, z)$ the domain of definition of the function is a volume in xyz-space.

Example 3. Find the domain of definition of the function

$$z=\frac{1}{\sqrt{4-x^2-y^2}}.$$

Solution. The function has real values if $4-x^2-y^2>0$ or $x^2+y^2<4$. The latter inequality is satisfied by the coordinates of points lying inside a circle of radius 2 with centre at the coordinate origin. The domain of definition of the function is the interior of the circle (Fig 64).

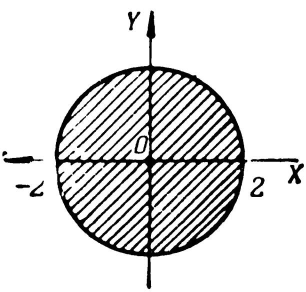

Fig. 64

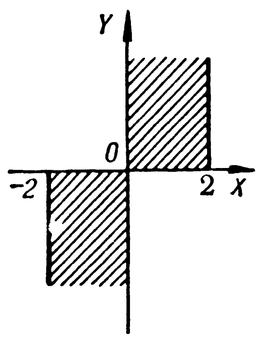

Fig 65

Example 4. Find the domain of definition of the function

$$z=\arcsin\frac{x}{2}+\sqrt{xy}$$

Solution. The first term of the function is defined for $-1\leqslant\frac{x}{2}\leqslant 1$ or $-2\leqslant x\leqslant 2$. The second term has real values if $xy\geqslant 0$, i.e., in two cases: when $\begin{cases} x\geqslant 0, \\ y\geqslant 0\end{cases}$ or when $\begin{cases} x\leqslant 0, \\ y\leqslant 0\end{cases}$. The domain of definition of the entire function is shown in Fig. 65 and includes the boundaries of the domain.

3°. Level lines and level surfaces of a function. The *level line* of a function $z=f(x, y)$ is a line $f(x, y)=C$ (in an xy-plane) at the points of which the function takes on one and the same value $z=C$ (usually labelled in drawings).

The *level surface* of a function of three arguments $u=f(x, y, z)$ is a surface $f(x, y, z)=C$, at the points of which the function takes on a constant value $u=C$.

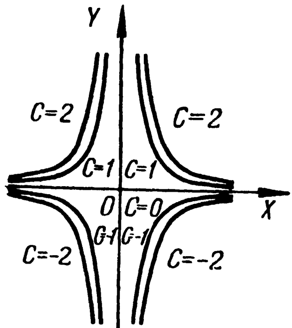

Fig. 66

Example 5. Construct the level lines of the function $z=x^2y$.

Solution. The equation of the level lines has the form $x^2y=C$ or $y=\frac{C}{x^2}$.

Putting $C=0, \pm 1, \pm 2, \ldots$, we get a family of level lines (Fig. 66).

1782. Express the volume V of a regular tetragonal pyramid as a function of its altitude x and lateral edge y.

1783. Express the lateral surface S of a regular hexagonal truncated pyramid as a function of the sides x and y of the bases and the altitude z.

1784. Find $f(1/2, 3)$, $f(1, -1)$, if

$$f(x, y)=xy+\frac{x}{y}.$$

1785 Find $f(y,x)$, $f(-x, -y)$, $f\left(\frac{1}{x}, \frac{1}{y}\right)$, $\frac{1}{f(x, y)}$, if $f(x, y)=\frac{x^2-y^2}{2xy}$.

1786. Find the values assumed by the function

$$f(x, y)=1+x-y$$

at points of the parabola $y=x^2$, and construct the graph of the function

$$F(x)=f(x, x^2).$$

1787. Find the value of the function

$$z=\frac{x^4+2x^2y^2+y^4}{1-x^2-y^2}$$

at points of the circle $x^2+y^2=R^2$.

1788*. Determine $f(x)$, if

$$f\left(\frac{y}{x}\right)=\frac{\sqrt{x^2+y^2}}{y} \quad (y>0).$$

1789*. Find $f(x, y)$ if

$$f(x+y, x-y)=xy+y^2.$$

1790*. Let $z=\sqrt{y}+f(\sqrt{x}-1)$. Determine the functions f and z if $z=x$ when $y=1$.

1791**. Let $z=xf\left(\frac{y}{x}\right)$. Determine the functions f and z if $z=\sqrt{1+y^2}$ when $x=1$.

1792. Find and sketch the domains of definition of the following functions:

a) $z=\sqrt{1-x^2-y^2}$;
b) $z=1+\sqrt{-(x-y)^2}$;
c) $z=\ln(x+y)$;
d) $z=x+\arccos y$;
e) $z=\sqrt{1-x^2}+\sqrt{1-y^2}$;
f) $z=\arcsin\frac{y}{x}$;
g) $z=\sqrt{x^2-4}+\sqrt{4-y^2}$;
h) $z=\sqrt{(x^2+y^2-a^2)(2a^2-x^2-y^2)}$ $(a>0)$;
i) $z=\sqrt{y\sin x}$;
j) $z=\ln(x^2+y)$;
k) $z=\operatorname{arc\,tang}\frac{x-y}{1+x^2y^2}$;
l) $z=\frac{1}{x^2+y^2}$;
m) $z=\frac{1}{\sqrt{y-\sqrt{x}}}$;
n) $z=\frac{1}{x-1}+\frac{1}{y}$;
o) $z\sqrt{\sin(x^2+y^2)}$.

1793. Find the domains of the following functions of three arguments:

a) $u=\sqrt{x}+\sqrt{y}+\sqrt{z}$;
b) $u=\ln(xyz)$;
c) $u=\arcsin x+\arcsin y+\arcsin z$;
d) $u=\sqrt{1-x^2-y^2-z^2}$.

1794. Construct the level lines of the given functions and determine the character of the surfaces depicted by these functions:

a) $z=x+y$;
b) $z=x^2+y^2$;
c) $z=x^2-y^2$;
d) $z=\sqrt{xy}$;
e) $z=(1+x+y)^2$;
f) $z=1-|x|-|y|$;
g) $z=\frac{y}{x^2}$;
h) $z=\frac{y}{\sqrt{x}}$;
i) $z=\frac{2x}{x^2+y^2}$.

1795. Find the level lines of the following functions:

a) $z=\ln(x^2+y)$;
b) $z=\arcsin xy$;
c) $z=f(\sqrt{x^2+y^2})$;
d) $z=f(y-ax)$;
e) $z=f\left(\frac{y}{x}\right)$.

1796. Find the level surfaces of the functions of three independent variables:

a) $u=x+y+z$;
b) $u=x^2+y^2+z^2$;
c) $u=x^2+y^2-z^2$.

Sec. 2. Continuity

1°. The limit of a function. A number A is called the *limit of a function* $z=f(x, y)$ as the point $P'(x, y)$ approaches the point $P(a, b)$, if for any $\varepsilon>0$ there is a $\delta>0$ such that when $0<\rho<\delta$, where $\rho=\sqrt{(x-a)^2+(y-b)^2}$ is the distance between P and P', we have the inequality

$$|f(x, y)-A|<\varepsilon.$$

In this case we write

$$\lim_{\substack{x\to a\\ y\to b}} f(x, y)=A.$$

2°. Continuity and points of discontinuity. A function $z=f(x, y)$ is called *continuous at a point* $P(a, b)$ if

$$\lim_{\substack{x\to a\\ y\to b}} f(x, y)=f(a, b).$$

A function that is continuous at all points of a given range is called *continuous over this range*

A function $f(x, y)$ may cease to be continuous either at separate points (*isolated point of discontinuity*) or at points that form one or several lines (*lines of discontinuity*) or (at times) more complex geometric objects.

Example 1. Find the discontinuities of the function

$$z=\frac{xy+1}{x^2-y}.$$

Solution. The function will be meaningless if the denominator becomes zero. But $x^2-y=0$ or $y=x^2$ is the equation of a parabola. Hence, the given function has for its discontinuity the parabola $y=x^2$.

1797*. Find the following limits of functions:

a) $\lim\limits_{\substack{x\to 0\\ y\to 0}} (x^2+y^2)\sin\frac{1}{xy}$; c) $\lim\limits_{\substack{x\to 0\\ y\to 2}} \frac{\sin xy}{x}$; e) $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{x}{x+y}$;

b) $\lim\limits_{\substack{x\to \infty\\ y\to \infty}} \frac{x+y}{x^2+y^2}$; d) $\lim\limits_{\substack{x\to \infty\\ y\to k}} \left(1+\frac{y}{x}\right)^x$; f) $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{x^2-y^2}{x^2+y^2}$.

1798. Test the following function for continuity:

$$f(x, y)=\begin{cases}\sqrt{1-x^2-y^2} & \text{when } x^2+y^2\leqslant 1,\\ 0 & \text{when } x^2+y^2>1.\end{cases}$$

1799. Find points of discontinuity of the functions:

a) $z=\ln\sqrt{x^2+y^2}$; c) $z=\frac{1}{1-x^2-y^2}$;

b) $z=\frac{1}{(x-y)^2}$; d) $z=\cos\frac{1}{xy}$.

1800*. Show that the function

$$z=\begin{cases}\dfrac{2xy}{x^2+y^2} & \text{when } x^2+y^2\neq 0,\\ 0 & \text{when } x=y=0\end{cases}$$

is continuous with respect to each of the variables x and y separately, but is not continuous at the point (0, 0) with respect to these variables together.

Sec. 3. Partial Derivatives

1°. Definition of a partial derivative. If $z=f(x, y)$, then assuming, for example, y constant, we get the derivative

$$\frac{\partial z}{\partial x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x, y)-f(x, y)}{\Delta x}=f'_x(x, y),$$

which is called the *partial derivative* of the function z with respect to the variable x. In similar fashion we define and denote the partial derivative of the function z with respect to the variable y It is obvious that to find partial derivatives, one can use the ordinary formulas of differentiation.

Example 1. Find the partial derivatives of the function

$$z=\ln\tan\frac{x}{y}.$$

Solution. Regarding y as constant, we get

$$\frac{\partial z}{\partial x}=\frac{1}{\tan\frac{x}{y}}\,\frac{1}{\cos^2\frac{x}{y}}\cdot\frac{1}{y}=\frac{2}{y\sin\frac{2x}{y}}.$$

Similarly, holding x constant, we will have

$$\frac{\partial z}{\partial y}=\frac{1}{\tan\frac{x}{y}}\cdot\frac{1}{\cos^2\frac{x}{y}}\left(-\frac{x}{y^2}\right)=-\frac{2x}{y^2\sin\frac{2x}{y}}.$$

Example 2. Find the partial derivatives of the following function of three arguments:

$$u=x^3y^2z+2x-3y+z+5.$$

Solution.

$$\frac{\partial u}{\partial x}=3x^2y^2z+2,$$

$$\frac{\partial u}{\partial y}=2x^3yz-3,$$

$$\frac{\partial u}{\partial z}=x^3y^2+1.$$

2°. Euler's theorem. A function $f(x, y)$ is called a *homogeneous* function of degree n if for every real factor k we have the equality

$$f(kx, ky)=k^nf(x, y)$$

A rational integral function will be homogeneous if all its terms are of one and the same degree.

The following relationship holds for a homogeneous differentiable function of degree n (*Euler's theorem*):

$$xf'_x(x, y)+yf'_y(x, y)=nf(x, y).$$

Find the partial derivatives of the following functions:

1801. $z=x^3+y^3-3axy$.

1802. $z=\frac{x-y}{x+y}$.

1803. $z=\frac{y}{x}$.

1804. $z=\sqrt{x^2-y^2}$.

1805. $z=\frac{x}{\sqrt{x^2+y^2}}$.

1806. $z=\ln(x+\sqrt{x^2+y^2})$.

1807. $z=\arctan\frac{y}{x}$.

1808. $z=x^y$.

1809. $z=e^{\sin\frac{y}{x}}$.

1810. $z=\arcsin\sqrt{\frac{x^2-y^2}{x^2+y^2}}$.

1811. $z=\ln\sin\frac{x+a}{\sqrt{y}}$.

1812. $u=(xy)^z$.

1813. $u=z^{xy}$.

1814. Find $f'_x(2, 1)$ and $f'_y(2, 1)$ if $f(x, y)=\sqrt{xy+\frac{x}{y}}$.

1815. Find $f'_x(1, 2, 0)$, $f'_y(1, 2, 0)$, $f'_z(1, 2, 0)$ if

$$f(x, y, z)=\ln(xy+z).$$

Verify Euler's theorem on homogeneous functions in Examples 1816 to 1819:

1816. $f(x, y)=Ax^2+2Bxy-Cy^2$.

1817. $z=\frac{x}{x^2+y^2}$.

1818. $f(x, y)=\frac{x+y}{\sqrt[3]{x^2+y^2}}$.

1819. $f(x, y)=\ln\frac{y}{x}$.

1820. Find $\frac{\partial}{\partial x}\left(\frac{1}{r}\right)$, where $r=\sqrt{x^2+y^2+z^2}$.

1821. Calculate $\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{vmatrix}$, if $x=r\cos\varphi$ and $y=r\sin\varphi$.

1822. Show that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2$, if $z=\ln(x^2+xy+y^2)$.

1823. Show that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=xy+z$, if $z=xy+xe^{\frac{y}{x}}$.

1824. Show that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$, if $u=(x-y)(y-z)(z-x)$.

1825. Show that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=1$, if $u=x+\frac{x-y}{y-z}$.

1826. Find $z=z(x, y)$, if $\frac{\partial z}{\partial y}=\frac{x}{x^2+y^2}$.

1827. Find $z=z(x, y)$ knowing that

$$\frac{\partial z}{\partial x}=\frac{x^2+y^2}{x} \text{ and } z(x, y)=\sin y \text{ when } x=1.$$

1828. Through the point $M(1, 2, 6)$ of a surface $z=2x^2+y^2$ are drawn planes parallel to the coordinate surfaces XOZ and YOZ. Determine the angles formed with the coordinate axes by the tangent lines (to the resulting cross-sections) drawn at their common point M.

1829. The area of a trapezoid with bases a and b and altitude h is equal to $S=\frac{1}{2}(a+b)h$. Find $\frac{\partial S}{\partial a}, \frac{\partial S}{\partial b}, \frac{\partial S}{\partial h}$ and, using the drawing, determine their geometrical meaning.

1830*. Show that the function

$$f(x, y)=\begin{cases} \dfrac{2xy}{x^2+y^2}, & \text{if } x^2+y^2\neq 0, \\ 0, & \text{if } x=y=0, \end{cases}$$

has partial derivatives $f'_x(x, y)$ and $f'_y(x, y)$ at the point (0, 0), although it is discontinuous at this point. Construct the geometric image of this function near the point (0, 0).

Sec. 4. Total Differential of a Function

1°. **Total increment of a function.** The *total increment* of a function $z=f(x, y)$ is the difference

$$\Delta z=\Delta f(x, y)=f(x+\Delta x, y+\Delta y)-f(x, y).$$

2°. **The total differential of a function.** The *total* (or *exact*) *differential* of a function $z=f(x, y)$ is the principal part of the total increment Δz, which is linear with respect to the increments in the arguments Δx and Δy.

The difference between the total increment and the total differential of the function is an infinitesimal of higher order compared with $\rho=\sqrt{\Delta x^2+\Delta y^2}$.

A function definitely has a total differential if its partial derivatives are continuous. If a function has a total differential, then it is called *differentiable*. The differentials of independent variables coincide with their increments, that is, $dx=\Delta x$ and $dy=\Delta y$. The total differential of the function $z=f(x, y)$ is computed by the formula

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy.$$

Similarly, the total differential of a function of three arguments $u=f(x, y, z)$ is computed from the formula

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz.$$

Example 1. For the function

$$f(x, y)=x^2+xy-y^2$$

find the total increment and the total differential.

Solution. $f(x+\Delta x,\ y+\Delta y)=(x+\Delta x)^2+(x+\Delta x)(y+\Delta y)-(y+\Delta y)^2;$

$$\begin{aligned}\Delta f(x,\ y)&=[(x+\Delta x)^2+(x+\Delta x)(y+\Delta y)-(y+\Delta y)^2]-(x^2+xy-y^2)=\\&=2x\cdot\Delta x+\Delta x^2+x\cdot\Delta y+y\cdot\Delta x+\Delta x\cdot\Delta y-2y\cdot\Delta y-\Delta y^2=\\&=[(2x+y)\,\Delta x+(x-2y)\,\Delta y]+(\Delta x^2+\Delta x\cdot\Delta y-\Delta y^2).\end{aligned}$$

Here, the expression $df=(2x+y)\,\Delta x+(x-2y)\,\Delta y$ is the total differential of the function, while $(\Delta x^2+\Delta x\cdot\Delta y-\Delta y^2)$ is an infinitesimal of higher order compared with $\sqrt{\Delta x^2+\Delta y^2}$.

Example 2. Find the total differential of the function

$$z=\sqrt{x^2+y^2}.$$

Solution. $\dfrac{\partial z}{\partial x}=\dfrac{x}{\sqrt{x^2+y^2}};\quad \dfrac{\partial z}{\partial y}=\dfrac{y}{\sqrt{x^2+y^2}}.$

$$dz=\frac{x}{\sqrt{x^2+y^2}}\,dx+\frac{y}{\sqrt{x^2+y^2}}\,dy=\frac{x\,dx+y\,dy}{\sqrt{x^2+y^2}}.$$

3°. Applying the total differential of a function to approximate calculations. For sufficiently small $|\Delta x|$ and $|\Delta y|$ and, hence, for sufficiently small $\varrho=\sqrt{\Delta x^2+\Delta y^2}$, we have for a differentiable function $z=f(x,\ y)$ the approximate equality $\Delta z\approx dz$ or

$$\Delta z\approx\frac{\partial z}{\partial x}\,\Delta x+\frac{\partial z}{\partial y}\,\Delta y.$$

Example 3. The altitude of a cone is $H=30$ cm, the radius of the base $R=10$ cm. How will the volume of the cone change, if we increase H by 3 mm and diminish R by 1 mm?

Solution. The volume of the cone is $V=\frac{1}{3}\pi R^2H$. The change in volume we replace approximately by the differential

$$\begin{aligned}\Delta V\approx dV&=\frac{1}{3}\pi\,(2RH\,dR+R^2\,dH)=\\&=\frac{1}{3}\pi\,(-2\cdot10\cdot30\cdot0.1+100\cdot0.3)=-10\pi\approx-31.4\ \text{cm}^3.\end{aligned}$$

Example 4. Compute $1.02^{3.01}$ approximately.

Solution. We consider the function $z=x^y$. The desired number may be considered the increased value of this function when $x=1$, $y=3$, $\Delta x=0.02$, $\Delta y=0.01$. The initial value of the function $z=1^3=1$,

$$\Delta z\approx dz=yx^{y-1}\,\Delta x+x^y\ln x\,\Delta y=3\cdot1\cdot0.02+1\cdot\ln1\cdot0.01=0.06.$$

Hence, $1.02^{3.01}\approx1+0.06=1.06$.

1831. For the function $f(x,\ y)=x^2y$ find the total increment and the total differential at the point $(1,\ 2)$; compare them if

a) $\Delta x=1$, $\Delta y=2$; b) $\Delta x=0.1$, $\Delta y=0.2$.

1832. Show that for the functions u and v of several (for example, two) variables the ordinary rules of differentiation hold:

a) $d(u+v)=du+dv$; b) $d(uv)=u\,dv+v\,du$;

c) $d\left(\dfrac{u}{v}\right)=\dfrac{v\,du-u\,dv}{v^2}$.

Find the total differentials of the following functions:

1833. $z=x^3+y^3-3xy$.

1834. $z=x^2y^3$.

1835. $z=\frac{x^2-y^2}{x^2+y^2}$.

1836. $z=\sin^2 x+\cos^2 y$.

1837. $z=yx^y$.

1838. $z=\ln(x^2+y^2)$.

1839. $f(x, y)=\ln\left(1+\frac{x}{y}\right)$.

1840. $z=\arc\tan\frac{y}{x}+$ $+\arc\tan\frac{x}{y}$.

1841. $z=\ln\tan\frac{y}{x}$.

1842. Find $df(1, 1)$, if $f(x, y)=\frac{x}{y^2}$.

1843. $u=xyz$.

1844. $u=\sqrt{x^2+y^2+z^2}$.

1845. $u=\left(xy+\frac{x}{y}\right)^z$.

1846. $u=\arc\tan\frac{xy}{z^2}$.

1847. Find $df(3, 4, 5)$ if $f(x, y, z)=\frac{z}{\sqrt{x^2+y^2}}$.

1848. One side of a rectangle is $a=10$ cm, the other $b=24$ cm. How will a diagonal l of the rectangle change if the side a is increased by 4 mm and b is shortened by 1 mm? Approximate the change and compare it with the exact value.

1849. A closed box with outer dimensions 10 cm, 8 cm, and 6 cm is made of 2-mm-thick plywood. Approximate the volume of material used in making the box.

1850*. The central angle of a circular sector is 80°; it is desired to reduce it by 1°. By how much should the radius of the sector be increased so that the area will remain unchanged, if the original length of the radius is 20 cm?

1851. Approximate:

a) $(1.02)^3 \cdot (0.97)^2$; b) $\sqrt{(4.05)^2+(2.93)^2}$;

c) $\sin 32^\circ \cdot \cos 59^\circ$ (when converting degrees into radius and calculating $\sin 60^\circ$ take three significant figures; round off the last digit).

1852. Show that the relative error of a product is approximately equal to the sum of the relative errors of the factors.

1853. Measurements of a triangle ABC yielded the following data: side $a=100\text{m}\pm 2\text{m}$. side $b=200\text{m}\pm 3$ m, angle $C=60^\circ\pm 1^\circ$. To what degree of accuracy can we compute the side c?

1854. The oscillation period T of a pendulum is computed from the formula

$$T=2\pi\sqrt{\frac{l}{g}},$$

where l is the length of the pendulum and g is the acceleration of gravity. Find the error, when determining T, obtained as a result of small errors $\Delta l=\alpha$ and $\Delta g=\beta$ in measuring l and g.

1855. The distance between the points $P_0(x_0, y_0)$ and $P(x, y)$ is equal to ϱ, while the angle formed by the vector $\overline{P_0P}$ with the x-axis is α. By how much will the angle α change if the point P (P_0 is fixed) moves to $P_1(x+dx, y+dy)$?

Sec. 5. Differentiation of Composite Functions

1°. The case of one independent variable. If $z=f(x, y)$ is a differentiable function of the arguments x and y, which in turn are differentiable functions of an independent variable t,

$$x=\varphi(t), \quad y=\psi(t),$$

then the derivative of the composite function $z=f[\varphi(t), \psi(t)]$ may be computed from the formula

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}. \tag{1}$$

In particular, if t coincides with one of the arguments, for instance x, then the "total" derivative of the function z with respect to x will be:

$$\frac{dz}{dx}=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\frac{dy}{dx}. \tag{2}$$

Example 1. Find $\frac{dz}{dt}$, if

$$z=e^{3x+2y}, \text{ where } x=\cos t, \ y=t^2.$$

Solution. From formula (1) we have:

$$\frac{dz}{dt}=e^{3x+2y}\cdot 3(-\sin t)+e^{3x+2y}\cdot 2\cdot 2t=e^{3x+2y}(4t-3\sin t)=e^{3\cos t+2t^2}(4t-3\sin t).$$

Example 2. Find the partial derivative $\frac{\partial z}{\partial x}$ and the total derivative $\frac{dz}{dx}$, if

$$z=e^{xy}, \text{ where } y=\varphi(x).$$

Solution. $\frac{\partial z}{\partial x}=ye^{xy}$.

From formula (2) we obtain

$$\frac{dz}{dx}=ye^{xy}+xe^{xy}\varphi'(x).$$

2°. The case of several independent variables. If z is a composite function of several independent variables, for instance, $z=f(x,y)$, where $x=\varphi(u,v)$, $y=\psi(u, v)$ (u and v are independent variables), then the partial derivatives z with respect to u and v are expressed as

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \tag{3}$$

and

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}\,. \tag{4}$$

In all the cases considered the following formula holds:

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$$

(the invariance property of a total differential).

Example 3. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$, if

$$z=f(x,\ y),\quad \text{where}\quad x=uv,\ y=\frac{u}{v}\,.$$

Solution. Applying formulas (3) and (4), we get:

$$\frac{\partial z}{\partial u}=f'_x(x,\ y)\cdot v+f'_y(x,\ y)\frac{1}{v}$$

and

$$\frac{\partial z}{\partial v}=f'_x(x,\ y)\ u-f'_y(x,\ y)\frac{u}{v^2}\,.$$

Example 4. Show that the function $z=\varphi(x^2+y^2)$ satisfies the equation

$$y\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}=0.$$

Solution. The function φ depends on x and y via the intermediate argument $x^2+y^2=t$, therefore,

$$\frac{\partial z}{\partial x}=\frac{dz}{dt}\frac{\partial t}{\partial x}=\varphi'(x^2+y^2)\,2x$$

and

$$\frac{\partial z}{\partial y}=\frac{dz}{dt}\frac{\partial t}{\partial y}=\varphi'(x^2+y^2)\,2y\,.$$

Substituting the partial derivatives into the left-hand side of the equation, we get

$$y\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}=y\varphi'(x^2+y^2)\,2x-x\varphi'(x^2+y^2)\,2y=2xy\,\varphi'(x^2+y^2)-2xy\,\varphi'(x^2+y^2)\equiv 0,$$

that is, the function z satisfies the given equation.

1856. Find $\frac{dz}{dt}$ if

$$z=\frac{x}{y},\ \text{where}\ x=e^t,\ \ y=\ln t.$$

1857. Find $\frac{du}{dt}$ if

$$u=\ln\sin\frac{x}{\sqrt{y}},\ \text{where}\ x=3t^2,\ \ y=\sqrt{t^2+1}.$$

1858. Find $\frac{du}{dt}$ if

$$u=xyz,\ \text{where}\ x=t^2+1,\ y=\ln t,\ z=\tan t.$$

1859. Find $\frac{du}{dt}$ if

$$u=\frac{z}{\sqrt{x^2+y^2}}, \text{ where } x=R\cos t,\ y=R\sin t,\ z=H.$$

1860. Find $\frac{dz}{dx}$ if

$$z=u^v, \text{ where } u=\sin x,\ v=\cos x.$$

1861. Find $\frac{\partial z}{\partial x}$ and $\frac{dz}{dx}$ if

$$z=\operatorname{arc\,tan}\frac{y}{x} \text{ and } y=x^2.$$

1862. Find $\frac{\partial z}{\partial x}$ and $\frac{dz}{dx}$ if

$$z=x^y, \text{ where } y=\varphi(x).$$

1863. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if

$$z=f(u,\ v), \text{ where } u=x^2-y^2,\ v=e^{xy}.$$

1864. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ if

$$z=\operatorname{arc\,tan}\frac{x}{y}, \text{ where } x=u\sin v,\ y=u\cos v.$$

1865. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if

$$z=f(u), \text{ where } u=xy+\frac{y}{x}.$$

1866. Show that if

$$u=\Phi(x^2+y^2+z^2), \text{ where } x=R\cos\varphi\cos\psi,$$
$$y=R\cos\varphi\sin\psi,\quad z=R\sin\varphi,$$

then

$$\frac{\partial u}{\partial\varphi}=0 \text{ and } \frac{\partial u}{\partial\psi}=0.$$

1867. Find $\frac{du}{dx}$ if

$$u=f(x,\ y,\ z), \text{ where } y=\varphi(x),\ z=\psi(x,\ y).$$

1868. Show that if

$$z=f(x+ay),$$

where f is a differentiable function, then

$$\frac{\partial z}{\partial y}=a\frac{\partial z}{\partial x}.$$

1869. Show that the function

$$w=f(u, v),$$

where $u=x+at$, $v=y+bt$ satisfy the equation

$$\frac{\partial w}{\partial t}=a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}.$$

1870. Show that the function

$$z=y\varphi(x^2-y^2)$$

satisfies the equation $\frac{1}{x}\frac{\partial z}{\partial x}+\frac{1}{y}\frac{\partial z}{\partial y}=\frac{z}{y^2}$.

1871. Show that the function

$$z=xy+x\varphi\left(\frac{y}{x}\right)$$

satisfies the equation $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=xy+z$.

1872. Show that the function

$$z=e^y\varphi\left(ye^{\frac{x^2}{2y^2}}\right)$$

satisfies the equation $(x^2-y^2)\frac{\partial z}{\partial x}+xy\frac{\partial z}{\partial y}=xyz$.

1873. The side of a rectangle $x=20$ m increases at the rate of 5 m/sec, the other side $y=30$ m decreases at 4 m/sec. What is the rate of change of the perimeter and the area of the rectangle?

1874. The equations of motion of a material point are

$$x=t,\ y=t^2,\ z=t^3.$$

What is the rate of recession of this point from the coordinate origin?

1875. Two boats start out from A at one time; one moves northwards, the other in a northeasterly direction. Their velocities are respectively 20 km/hr and 40 km/hr. At what rate does the distance between them increase?

Sec. 6. Derivative in a Given Direction and the Gradient of a Function

1°. The derivative of a function in a given direction. The *derivative* of a function $z=f(x, y)$ *in a given direction* $l=\overrightarrow{PP_1}$ is

$$\frac{\partial z}{\partial l}=\lim_{P_1P\to 0}\frac{f(P_1)-f(P)}{P_1P},$$

where $f(P)$ and $f(P_1)$ are values of the function at the points P and P_1. If the function z is differentiable, then the following formula holds:

$$\frac{\partial z}{\partial l}=\frac{\partial z}{\partial x}\cos\alpha+\frac{\partial z}{\partial y}\sin\alpha, \tag{1}$$

where α is the angle formed by the vector l with the x-axis (Fig. 67).

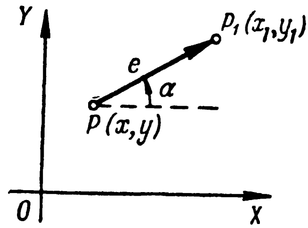

Fig. 67

In similar fashion we define the derivative in a given direction l for a function of three arguments $u=f(x, y, z)$. In this case

$$\frac{\partial u}{\partial l}=\frac{\partial u}{\partial x}\cos\alpha+\frac{\partial u}{\partial y}\cos\beta+\frac{\partial u}{\partial z}\cos\gamma, \tag{2}$$

where α, β, γ are the angles between the direction l and the corresponding coordinate axes. The directional derivative characterises the rate of change of the function in the given direction.

Example 1. Find the derivative of the function $z=2x^2-3y^2$ at the point $P(1, 0)$ in a direction that makes a 120° angle with the x-axis.

Solution. Find the partial derivatives of the given function and their values at the point P:

$$\frac{\partial z}{\partial x}=4x;\quad \left(\frac{\partial z}{\partial x}\right)_P=4;$$

$$\frac{\partial z}{\partial y}=-6y;\quad \left(\frac{\partial z}{\partial y}\right)_P=0.$$

Here,

$$\cos\alpha=\cos 120^\circ=-\frac{1}{2},$$

$$\sin\alpha=\sin 120^\circ=\frac{\sqrt{3}}{2}.$$

Applying formula (1), we get

$$\frac{\partial z}{\partial l}=4\left(-\frac{1}{2}\right)+0\cdot\frac{\sqrt{3}}{2}=-2.$$

The minus sign indicates that the function diminishes at the given point and in the given direction.

2°. **The gradient of a function.** The *gradient* of a function $z=f(x, y)$ is a vector whose projections on the coordinate axes are the corresponding par-

tial derivatives of the given function:

$$\text{grad } z=\frac{\partial z}{\partial x}\boldsymbol{i}+\frac{\partial z}{\partial y}\boldsymbol{j}. \tag{3}$$

The derivative of the given function in the direction $\boldsymbol{l}$ is connected with the gradient of the function by the following formula:

$$\frac{\partial z}{\partial l}=\text{pro}\boldsymbol{j_l}\,\text{grad } z.$$

That is, the derivative in a given direction is equal to the projection of the gradient of the function on the direction of differentiation.

The gradient of a function at each point is directed along the normal to the corresponding level line of the function. The direction of the gradient of the function at a given point is the direction of the maximum rate of increase of the function at this point, that is, when $\boldsymbol{l}=\text{grad } z$ the derivative $\frac{\partial z}{\partial l}$ takes on its greatest value, equal to

$$\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}.$$

In similar fashion we define the gradient of a function of three variables, $u=f(x, y, z)$:

$$\text{grad } u=\frac{\partial u}{\partial x}\boldsymbol{i}+\frac{\partial u}{\partial y}\boldsymbol{j}+\frac{\partial u}{\partial z}\boldsymbol{k}. \tag{4}$$

The gradient of a function of three variables at each point is directed along the normal to the level surface passing through this point.

Example 2. Find and construct the gradient of the function $z=x^2y$ at the point $P(1, 1)$.

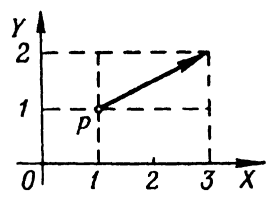

Fig. 68

Solution. Compute the partial derivatives and their values at the point P.

$$\frac{\partial z}{\partial x}=2xy; \quad \left(\frac{\partial z}{\partial x}\right)_P=2;$$

$$\frac{\partial z}{\partial y}=x^2; \quad \left(\frac{\partial z}{\partial y}\right)_P=1.$$

Hence, grad $z=2\boldsymbol{i}+\boldsymbol{j}$ (Fig. 68).

1876. Find the derivative of the function $z=x^2-xy-2y^2$ at the point $P(1, 2)$ in the direction that produces an angle of 60° with the x-axis.

1877. Find the derivative of the function $z=x^3-2x^2y+xy^2+1$ at the point $M(1, 2)$ in the direction from this point to the point $N(4, 6)$.

1878. Find the derivative of the function $z=\ln\sqrt{x^2+y^2}$ at the point $P(1, 1)$ in the direction of the bisector of the first quadrantal angle.

1879. Find the derivative of the function $u=x^2-3yz+5$ at the point $M(1, 2, -1)$ in the direction that forms identical angles with all the coordinate axes.

1880. Find the derivative of the function $u=xy+yz+zx$ at the point $M(2, 1, 3)$ in the direction from this point to the point $N(5, 5, 15)$.

1881. Find the derivative of the function $u=\ln(e^x+e^y+e^z)$ at the origin in the direction which forms with the coordinate axes x, y, z the angles α, β, γ, respectively.

1882. The point at which the derivative of a function in any direction is zero is called the *stationary point* of this function. Find the stationary points of the following functions:

a) $z=x^2+xy+y^2-4x-2y$;

b) $z=x^3+y^3-3xy$;

c) $u=2y^2+z^2-xy-yz+2x$.

1883. Show that the derivative of the function $z=\frac{y^2}{x}$ taken at any point of the ellipse $2x^2+y^2=C^2$ along the normal to the ellipse is equal to zero.

1884. Find grad z at the point (2, 1) if

$$z=x^3+y^3-3xy.$$

1885. Find grad z at the point (5, 3) if

$$z=\sqrt{x^2-y^2}.$$

1886. Find grad u at the point (1, 2, 3), if $u=xyz$.

1887. Find the magnitude and direction of grad u at the point (2, —2, 1) if

$$u=x^2+y^2+z^2.$$

1888. Find the angle between the gradients of the function $z=\ln\frac{y}{x}$ at the points $A(1/2, 1/4)$ and $B(1, 1)$.

1889. Find the steepest slope of the surface

$$z=x^2+4y^2$$

at the point (2, 1, 8).

1890. Construct a vector field of the gradient of the following functions:

a) $z=x+y$; c) $z=x^2+y^2$;

b) $z=xy$; d) $u=\dfrac{1}{\sqrt{x^2+y^2+z^2}}$.

Sec. 7. Higher-Order Derivatives and Differentials

1°. Higher-order partial derivatives. The *second partial derivatives* of a function $z=f(x, y)$ are the partial derivatives of its first partial derivatives.

For second derivatives we use the notations

$$\frac{\partial}{\partial x}\left(\frac{z}{\partial x}\right)=\frac{\partial^2 z}{\partial x^2}=f''_{xx}(x, y);$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x \partial y}=f''_{xy}(x, y) \text{ and so forth.}$$

Derivatives of order higher than second are similarly defined and denoted.

If the partial derivatives to be evaluated are continuous, then the *result of repeated differentiation is independent of the order in which the differentiation is performed.*

Example 1. Find the second partial derivatives of the function

$$z=\arc\tan\frac{x}{y}.$$

Solution. First find the first partial derivatives:

$$\frac{\partial z}{\partial x}=\frac{1}{1+\frac{x^2}{y^2}}\cdot\frac{1}{y}=\frac{y}{x^2+y^2},$$

$$\frac{\partial z}{\partial y}=\frac{1}{1+\frac{x^2}{y^2}}\left(-\frac{x}{y^2}\right)=-\frac{x}{x^2+y^2}.$$

Now differentiate a second time:

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right)=-\frac{2xy}{(x^2+y^2)^2},$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(-\frac{x}{x^2+y^2}\right)=\frac{2xy}{(x^2+y^2)^2}.$$

$$\frac{\partial^2 z}{\partial x\,\partial y}=\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)=\frac{1\cdot(x^2+y^2)-2y\cdot y}{(x^2+y^2)^2}=\frac{x^2-y^2}{(x^2+y^2)^2}.$$

We note that the so-called "mixed" partial derivative may be found in a different way, namely:

$$\frac{\partial^2 z}{\partial x\,\partial y}=\frac{\partial^2 z}{\partial y\,\partial x}=\frac{\partial}{\partial x}\left(-\frac{x}{x^2+y^2}\right)=-\frac{1\cdot(x^2+y^2)-2x\cdot x}{(x^2+y^2)^2}=\frac{x^2-y^2}{(x^2+y^2)^2}.$$

2°. **Higher-order differentials.** The *second differential* of a function $z=f(x, y)$ is the differential of the differential (first-order) of this function:

$$d^2z=d(dz)$$

We similarly define the differentials of a function z of order higher than two, for instance:

$$d^3z=d(d^2z)$$

and, generally,

$$d^nz=d(d^{n-1}z).$$

If $z=f(x, y)$, where x and y are independent variables, then the second differential of the function z is computed from the formula

$$d^2z=\frac{\partial^2 z}{\partial x^2}dx^2+2\frac{\partial^2 z}{\partial x\,\partial y}dx\,dy+\frac{\partial^2 z}{\partial y^2}dy^2. \tag{1}$$

Generally, the following symbolic formula holds true:

$$d^nz=\left(dx\frac{\partial}{\partial x}+dy\frac{\partial}{\partial y}\right)^n z;$$

it is formally expanded by the binomial law.

If $z=f(x, y)$, where the arguments x and y are functions of one or several independent variables, then

$$d^2z=\frac{\partial^2 z}{\partial x^2}dx^2+2\frac{\partial^2 z}{\partial x\,\partial y}dx\,dy+\frac{\partial^2 z}{\partial y^2}dy^2+\frac{\partial z}{\partial x}d^2x+\frac{\partial z}{\partial y}d^2y. \tag{2}$$

If x and y are independent variables, then $d^2x=0$, $d^2y=0$, and formula (2) becomes identical with formula (1).

Example 2. Find the total differentials of the first and second orders of the function

$$z=2x^2-3xy-y^2.$$

Solution. First method. We have

$$\frac{\partial z}{\partial x}=4x-3y,\quad \frac{\partial z}{\partial y}=-3x-2y.$$

Therefore,

$$\partial z=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=(4x-3y)\,dx-(3x+2y)\,dy.$$

Further we have

$$\frac{\partial^2 z}{\partial x^2}=4,\quad \frac{\partial^2 z}{\partial x\,\partial y}=-3,\quad \frac{\partial^2 z}{\partial y^2}=-2,$$

whence it follows that

$$d^2z=\frac{\partial^2 z}{\partial x^2}dx^2+2\frac{\partial^2 z}{\partial x\,\partial y}dx\,dy+\frac{\partial^2 z}{\partial y^2}dy^2=4dx^2-6\,dx\,dy-2\,dy^2.$$

Second method. Differentiating we find

$$dz=4x\,dx-3\,(y\,dx+x\,dy)-2y\,dy=(4x-3y)\,dx-(3x+2y)\,dy.$$

Differentiating again and remembering that dx and dy are not dependent on x and y, we get

$$d^2z=(4dx-3dy)\,dx-(3dx+2dy)\,dy=4dx^2-6dx\,dy-2dy^2.$$

1891. Find $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x\,\partial y}$, $\frac{\partial^2 z}{\partial y^2}$ if

$$z = c\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}.$$

1892. Find $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x\,\partial y}$, $\frac{\partial^2 z}{\partial y^2}$ if

$$z = \ln(x^2+y).$$

1893. Find $\frac{\partial^2 z}{\partial x\,\partial y}$ if

$$z = \sqrt{2xy+y^2}.$$

1894. Find $\frac{\partial^2 z}{\partial x\,\partial y}$ if

$$z = \arctan\frac{x+y}{1-xy}.$$

1895. Find $\frac{\partial^2 r}{\partial x^2}$ if

$$r = \sqrt{x^2+y^2+z^2}.$$

1896. Find all second partial derivatives of the function

$$u = xy + yz + zx.$$

1897. Find $\frac{\partial^3 u}{\partial x\,\partial y\,\partial z}$ if

$$u = x^{\alpha}y^{\beta}z^{\gamma}.$$

1898. Find $\frac{\partial^3 z}{\partial x\partial y^2}$ if

$$z = \sin(xy).$$

1899. Find $f''_{xx}(0, 0)$, $f''_{xy}(0, 0)$, $f''_{yy}(0, 0)$ if

$$f(x, y) = (1+x)^m(1+y)^n.$$

1900. Show that $\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x}$ if

$$z = \arcsin\sqrt{\frac{x-y}{x}}.$$

1901. Show that $\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x}$ if

$$z = x^y.$$

1902*. Show that for the function

$$f(x, y) = xy\frac{x^2-y^2}{x^2+y^2}$$

[provided that $f(0, 0)=0$] we have

$$f''_{xy}(0, 0)=-1, \quad f''_{yx}(0, 0)=+1.$$

1903. Find $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x\, \partial y}, \frac{\partial^2 z}{\partial y^2}$ if

$$z=f(u, v),$$

where $u=x^2+y^2$, $v=xy$.

1904. Find $\frac{\partial^2 u}{\partial x^2}$ if $u=f(x, y, z)$, where $z=\varphi(x, y)$.

1905. Find $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x\, \partial y}, \frac{\partial^2 z}{\partial y^2}$ if

$$z=f(u, v), \text{ where } u=\varphi(x, y), \quad v=\psi(x, y).$$

1906. Show that the function

$$u=\arctan\frac{y}{x}$$

satisfies the Laplace equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

1907. Show that the function

$$u=\ln\frac{1}{r},$$

where $r=\sqrt{(x-a)^2+(y-b)^2}$, satisfies the Laplace equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

1908. Show that the function

$$u(x, t)=A\sin(a\lambda t+\varphi)\sin\lambda x$$

satisfies the equation of oscillations of a string

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}.$$

1909. Show that the function

$$u(x, y, z, t)=\frac{1}{(2a\sqrt{\pi t})^3}e^{-\frac{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}{4a^2t}}$$

(where x_0, y_0, z_0, a are constants) satisfies the equation of heat conduction

$$\frac{\partial u}{\partial t}=a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right).$$

1910. Show that the function

$$u=\varphi(x-at)+\psi(x+at),$$

where φ and ψ are arbitrary twice differentiable functions, satisfies the equation of oscillations of a string

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}.$$

1911. Show that the function

$$z=x\varphi\left(\frac{y}{x}\right)+\psi\left(\frac{y}{x}\right)$$

satisfies the equation

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\,\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=0.$$

1912. Show that the function

$$u=\varphi(xy)+\sqrt{xy}\,\psi\left(\frac{y}{x}\right)$$

satisfies the equation

$$x^2\frac{\partial^2 u}{\partial x^2}-y^2\frac{\partial^2 u}{\partial y^2}=0.$$

1913. Show that the function $z=f[x+\varphi(y)]$ satisfies the equation

$$\frac{\partial z}{\partial x}\frac{\partial^2 z}{\partial x\,\partial y}=\frac{\partial z}{\partial y}\frac{\partial^2 z}{\partial x^2}.$$

1914. Find $u=u(x, y)$ if

$$\frac{\partial^2 u}{\partial x\,\partial y}=0.$$

1915. Determine the form of the function $u=u(x, y)$, which satisfies the equation

$$\frac{\partial^2 u}{\partial x^2}=0.$$

1916. Find d^2z if

$$z=e^{xy}.$$

1917. Find d^2u if

$$u=xyz.$$

1918. Find d^2z if

$$z=\varphi(t),\ \text{where}\ t=x^2+y^2.$$

1919. Find dz and d^2z if

$$z=u^v\ \text{where}\ u=\frac{x}{y},\ v=xy.$$